

Geometric control theory

I: mathematical foundations *

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Abstract

A geometric setup for control theory is presented. The argument is developed through the study of the extremals of action functionals defined on piecewise differentiable curves, in the presence of differentiable non-holonomic constraints. Special emphasis is put on the tensorial aspects of the theory. To start with, the kinematical foundations, culminating in the so called *variational equation*, are put on geometrical grounds, via the introduction of the concept of *infinitesimal control*. On the same basis, the usual classification of the extremals of a variational problem into *normal* and *abnormal* ones is also rationalized, showing the existence of a purely kinematical algorithm assigning to each admissible curve a corresponding *abnormality index*, defined in terms of a suitable linear map. The whole machinery is then applied to constrained variational calculus. The argument provides an interesting revisitation of Pontryagin maximum principle and of the Erdmann–Weierstrass corner conditions, as well as a proof of the classical Lagrange multipliers method and a local interpretation of Pontryagin's equations as dynamical equations for a free (singular) Hamiltonian system. As a final, highly non-trivial topic, a sufficient condition for the existence of finite deformations with fixed endpoints is explicitly stated and proved.

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1 Introduction

Constrained variational calculus has been extensively studied since the beginning of the XX century and has been recently revived by its applications to control theory. Among others, we mention here the pioneering works of Bolza and Bliss [1], the contribution of Pontryagin [4] and the more recent developments by Sussman, Agrachev, Hsu, Montgomery and Griffiths [18, 21, 24, 20, 12], characterized by the use of the mathematical instruments provided by differential geometry.

As it is already clear in Bliss work, the primary task is finding out to what extent the classical tools for the determination of constrained extremals in ordinary Calculus (e.g. Lagrange multipliers) may be extended to the study of functionals.

Quite naturally, the argument requires a preliminary characterization of the admissible curves and of their deformations. In this connection, the analysis reveals the existence of curves having a pathological behaviour with respect the fixed-end deformations, an aspect well known in the literature, under the name of *abnormal extremals* [18, 20].

In this paper we propose a fresh approach to the argument. We shall deal with systems described by a finite number of variables q^1, \dots, q^n , subject to a set of differentiable conditions of the form

$$\frac{dq^i}{dt} = \psi^i(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad (1.1)$$

expressing the derivatives $\frac{dq^i}{dt}$ in terms of a smaller number of *control variables* z^A , $A = 1, \dots, r$. Every set of functions $q^i = q^i(t)$, $z^A = z^A(t)$ consistent with the requirement (1.1) will be called an *admissible evolution* of the system.

In addition to the constraints (1.2) we shall also consider an *action functional*

$$I = \int_{t_0}^{t_1} L(t, q^1, \dots, q^n, z^1, \dots, z^r) dt \quad (1.2)$$

expressed as the integral of a suitable “cost function”, or *Lagrangian* $L(t, q, z)$ along the admissible evolutions of the system. Within the stated context, we shall discuss the (local) extremals of I with respect to the admissible deformations leaving the endpoints $q^i(t_0)$, $q^i(t_1)$ fixed.

Geometrically, an intrinsic formulation of the problem is obtained introducing a fibre bundle $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, with coordinates t, q^1, \dots, q^n , called the *event space*, denoting by $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ the associated *first jet bundle*, and regarding the control equations (1.1) as the representation of a submanifold $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ describing the totality of admissible *kinetic states* of the system. In the resulting environment, a *control* for the system is then simply a section $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$, locally represented as $z^A = z^A(t, q^1, \dots, q^n)$.

The infinitesimal deformations of an admissible section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ are discussed via a revisitation of the familiar *variational equation*. The novelty of the approach relies on the introduction of a transport law for vertical vector fields along

γ , yielding a covariant characterization of the “true” degrees of freedom involved in the description of the most general admissible infinitesimal deformation.

The analysis is subsequently extended to arbitrary *piecewise differentiable* evolutions γ consisting of families of contiguous closed arcs $\gamma^{(s)} : [a_{s-1}, a_s] \rightarrow \mathcal{V}_{n+1}$, $t_0 = a_0 < a_1 < \dots < a_N = t_1$ denoting any (finite) partition of the interval $[t_0, t_1]$.

In this connection, special attention is paid to the infinitesimal deformations *vanishing at the endpoints*. No restrictions are posed on the deformability of the intervals $[a_{s-1}, a_s]$ or on the mobility of the “corners” $\gamma(a_s)$, $s = 1, \dots, N - 1$.

The argument allows to assign to every admissible evolution a corresponding *abnormality index*, rephrasing and putting on geometrical grounds the traditional attributes of normality and abnormality commonly found in the literature [18, 20].

Equally important, in a variational context, is finding out which infinitesimal deformations are actually tangent to finite deformations with fixed end points. To account for this aspect, the admissible evolutions will be classified into *ordinary* ones, if *every* admissible infinitesimal deformation vanishing at the endpoints is tangent to some finite deformations with fixed end points, and *exceptional* ones in the opposite case. An important result, proved in Appendix B and extending a result proved in [24] in the case of linear constraints, is the relationship between abnormality index and ordinariness, resulting in the fact that every *normal* evolutions is also, automatically, an ordinary one.

After these preliminaries, attention is focussed on the study of the extremals of the functional (1.2). Once again, the approach relies on a fully covariant algorithm, summarizing the content of Pontryagin’s maximum principle [4, 19] and of the Erdmann Weierstrass corner conditions [9, 19]. The resulting equations are shown to provide sufficient conditions for *any* evolution, and necessary and sufficient conditions for an *ordinary* evolution γ to be an extremal. The same setup is seen to provide an elegant and concise proof of the Lagrange multipliers method.

In the final part of the paper, the geometric content of the algorithm is further enhanced, lifting everything to a fiber bundle $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$, here called the *contact bundle*, defined as the pull-back of the *phase space* $V^*(\mathcal{V}_{n+1})$ through the fibered morphism

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \longrightarrow & V^*(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{V}_{n+1} \end{array}$$

In the resulting framework, the ordinary extremals of the original variational problem are seen to arise as projections of the solutions of a corresponding *free* variational problem on $\mathcal{C}(\mathcal{A})$. The Euler–Lagrange equations for the newer problem are written in terms of the so called Pontryagin Hamiltonian $\mathcal{H} := p_i \psi^i - L$, viewed as a function on $\mathcal{C}(\mathcal{A})$. The same environment provides also a characterization of the ordinary *abnormal* evolutions of the system as projections of the extremals of a purely geometric action principle based on the Liouville 1-form of $\mathcal{C}(\mathcal{A})$.

The argument is completed by a discussion of the meaning of the Pontryagin

Hamiltonian. It is seen that, under suitable regularity assumptions, the original constrained variational problem is locally equivalent to a free Hamiltonian problem on the phase space.

2 Geometric setup

2.1 Preliminaries

In this Section we outline the construction of a geometrical setup especially suited to the development of constrained variational calculus and smooth control theory. In this connection, see also [1, 2, 4, 5, 7, 8, 9, 12, 18, 19, 21] and references therein.

(i) Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote a fibre bundle over the real line, henceforth called the *event space*, and referred to local fibered coordinates t, q^1, \dots, q^n .

Every section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$, locally described as $q^i = q^i(t)$, will be interpreted as an *evolution* of an abstract system \mathfrak{B} , parameterized in terms of the independent variable t . The terminology is borrowed from mechanics, where \mathfrak{B} is identified with a material system, \mathcal{V}_{n+1} with the configuration space–time of \mathfrak{B} , and $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$ with the *absolute time* function.

Pursuing this analogy, the first jet bundle $j_1(\mathcal{V}_{n+1})$, referred to local jet coordinates t, q^i, \dot{q}^i , will be called the *velocity space*.

As reported in Appendix A, the projection $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ makes $j_1(\mathcal{V}_{n+1})$ into an affine bundle, modelled on the vertical space $V(\mathcal{V}_{n+1})$. The vertical bundle associated with this projection will be denoted by $V(j_1(\mathcal{V}_{n+1}))$.

Every section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ admits a lift $j_1(\gamma) : \mathbb{R} \rightarrow j_1(\mathcal{V}_{n+1})$, expressed in coordinates as $q^i = q^i(t), \dot{q}^i = \frac{dq^i}{dt}$. In a similar way, every vertical vector field $X = X^i \frac{\partial}{\partial q^i}$ over \mathcal{V}_{n+1} may be lifted to a field $J(X) = X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i}$ over $j_1(\mathcal{V}_{n+1})$, with $\dot{X}^i := \frac{\partial X^i}{\partial t} + \dot{q}^k \frac{\partial X^i}{\partial q^k}$. Both arguments are entirely standard (see e.g. [16]), and will be regarded as known.

(ii) The restriction of the vertical space $V(\mathcal{V}_{n+1})$ to the section γ determines a vector bundle $V(\gamma) \xrightarrow{t} \mathbb{R}$, called the vertical bundle over γ .

Proposition 2.1 *The first jet space $j_1(V(\gamma))$ is canonically isomorphic to the vector bundle over \mathbb{R} formed by the totality of vectors Z along $j_1(\gamma)$ satisfying $\langle Z, dt \rangle = 0$. With this identification, the fibration $\pi_* : j_1(V(\gamma)) \rightarrow V(\gamma)$ coincides with the restriction to $j_1(V(\gamma))$ of the push-forward of the projection $\pi : j_1(\mathcal{V}_{n+1}) \rightarrow \mathcal{V}_{n+1}$.*

Proof. Given $t^* \in \mathbb{R}$ and a section $X : \mathbb{R} \rightarrow V(\gamma)$, choose any vector field \tilde{X} defined in a neighborhood $U \ni \gamma(t^*)$ and satisfying $\tilde{X}|_{\gamma(t)} = X(t) \ \forall t \in \gamma^{-1}(U)$.

In coordinates, setting $\gamma : q^i = q^i(t)$, $X = X^i(t) \left(\frac{\partial}{\partial q^i} \right)_\gamma$, the lift of the field \tilde{X} at the point $j_1(\gamma)(t^*)$ takes the value

$$J(\tilde{X})|_{j_1(\gamma)(t^*)} = X^i(t^*) \left(\frac{\partial}{\partial q^i} \right)_{j_1(\gamma)(t^*)} + \left. \frac{dX^i}{dt} \right|_{t=t^*} \left(\frac{\partial}{\partial \dot{q}^i} \right)_{j_1(\gamma)(t^*)} \quad (2.1)$$

All assertions of Proposition 2.1 follow immediately from this fact. \square

Consistently with eq. (2.1), given any section $X : \mathbb{R} \rightarrow V(\gamma)$, the jet extension $j_1(X)$ will be called the *lift* of X to the curve $j_1(\gamma)$. In local coordinates, eq. (2.1) provides the representation

$$j_1(X) = X^i \left(\frac{\partial}{\partial q^i} \right)_{j_1(\gamma)} + \frac{dX^i}{dt} \left(\frac{\partial}{\partial \dot{q}^i} \right)_{j_1(\gamma)} \quad (2.2)$$

Every local coordinate system t, q^i in \mathcal{V}_{n+1} determines fibered coordinates t, u^i in $V(\gamma)$ and t, u^i, \dot{u}^i in $j_1(V(\gamma))$, based on the identifications

$$X = u^i(X) \left(\frac{\partial}{\partial q^i} \right)_{\gamma(t(X))} \quad \forall X \in V(\gamma) \quad (2.3a)$$

$$Z = u^i(Z) \left(\frac{\partial}{\partial q^i} \right)_{j_1(\gamma)(t(Z))} + \dot{u}^i(Z) \left(\frac{\partial}{\partial \dot{q}^i} \right)_{j_1(\gamma)(t(Z))} \quad \forall Z \in j_1(V(\gamma)) \quad (2.3b)$$

In terms of these, the jet-extension of a section $u^i = u^i(t)$ takes the standard form $u^i = u^i(t)$, $\dot{u}^i = \frac{du^i}{dt}$, while the projection $\pi_* : j_1(V(\gamma)) \rightarrow V(\gamma)$ is described by $u^i(\pi_*(Z)) = u^i(Z)$. In particular, the vertical subbundle $V(j_1(\gamma))$ coincides with the submanifold of $j_1(V(\gamma))$ locally described by the equation $u^i = 0$, $i = 1, \dots, n$.

Corollary 2.1 *The vector bundles $V(j_1(\gamma)) \xrightarrow{t} \mathbb{R}$ and $V(\gamma) \xrightarrow{t} \mathbb{R}$ are canonically isomorphic*

Proof. As pointed out in Appendix A, for each $z \in j_1(\mathcal{V}_{n+1})$ the affine character of the fibration $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ determines an isomorphism between the vertical spaces $V_z(j_1(\mathcal{V}_{n+1}))$ and $V_{\pi(z)}(\mathcal{V}_{n+1})$, expressed in coordinates as

$$\varrho \left[W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_z \right] = W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_{\pi(z)}$$

In particular, for $z = j_1(\gamma)(t)$, our previous definitions imply the identifications $\pi(z) = \gamma(t)$, $V_{\pi(z)}(\mathcal{V}_{n+1}) = V(\gamma)|_t$, $V_z(j_1(\mathcal{V}_{n+1})) = V(j_1(\gamma))|_t$. By varying t , this gives rise to a vector bundle isomorphism

$$\begin{array}{ccc} V(j_1(\gamma)) & \xrightarrow{\varrho} & V(\gamma) \\ t \downarrow & & \downarrow t \\ \mathbb{R} & \xlongequal{\quad} & \mathbb{R} \end{array} \quad (2.4)$$

expressed in coordinates as

$$\varrho \left[W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_{j_1(\gamma)} \right] = W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_\gamma \quad (2.5)$$

□

(iii) Together with the vertical bundle $V(\gamma) \xrightarrow{t} \mathbb{R}$ it is worth considering the *dual* bundle $V^*(\gamma) \xrightarrow{t} \mathbb{R}$, identical to the pull-back on γ of the phase space $V^*(\mathcal{V}_{n+1})$. With the notation of Appendix A, the situation is expressed by the commutative diagram

$$\begin{array}{ccc} V^*(\gamma) & \longrightarrow & V^*(\mathcal{V}_{n+1}) \\ t \downarrow & & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\gamma} & \mathcal{V}_{n+1} \end{array} \quad (2.6)$$

The elements of $V^*(\gamma)$ will be called the *virtual 1-forms* along γ .

Notice that, according to the stated definition, a virtual 1-form $\hat{\lambda}$ at a point $\gamma(t)$ is *not* a 1-form in the ordinary sense, but an *equivalence class* of 1-forms under the relation

$$\lambda \sim \lambda' \quad \Longleftrightarrow \quad \lambda - \lambda' \propto (dt)_{\gamma(t)} \quad (2.7)$$

For simplicity, we preserve the notation $\langle \cdot, \cdot \rangle$ for the pairing between $V(\gamma)$ and $V^*(\gamma)$. Also, given any local coordinate system t, q^i in \mathcal{V}_{n+1} , we refer $V^*(\gamma)$ to fiber coordinates t, p_i , with $p_i(\hat{\lambda}) = \langle \hat{\lambda}, (\frac{\partial}{\partial q^i})_{\gamma(t(\hat{\lambda}))} \rangle$. The virtual 1-forms along γ determined by the differentials dq^i will be denoted by $\hat{\omega}^i$, $i = 1, \dots, n$.

For completeness we remark that, according to eq. (2.6), each fiber $V^*(\gamma)|_t$ is isomorphic to the subspace of the cotangent space $T_{\gamma(t)}^*(\mathcal{V}_{n+1})$ annihilating the tangent vector to the curve γ at the point $\gamma(t)$. Formally, this viewpoint is implemented by setting $\hat{\omega}^i = (dq^i - \frac{dq^i}{dt} dt)_{\gamma}$. Although apparently simpler, this characterization of $V^*(\gamma)$ has some drawbacks in the case of piecewise differentiable sections. We shall therefore stick to the original definition.

(iv) Let us now see how the geometric setup developed so far gets modified in the presence of non-holonomic constraints reducing the number of independent velocities. Under the explicit assumption of *differentiable constraints*, the situation is summarized into a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{V}_{n+1} & \xlongequal{\quad} & \mathcal{V}_{n+1} \end{array} \quad (2.8)$$

where

- $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a fiber bundle, representing the totality of *admissible velocities*;
- the map $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ is an imbedding;
- a section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ is admissible if and only if its lift $j_1(\gamma)$ factors through \mathcal{A} , i.e. if and only if there exists a section $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ satisfying $j_1(\gamma) = i \cdot \hat{\gamma}$.

The geometry of the submanifold \mathcal{A} is well known from non-holonomic mechanics. Concepts like the *vertical bundle* $V(\mathcal{A})$ and the *contact bundle* $\mathcal{C}(\mathcal{A})$ will be freely used in the following. Their properties are briefly reviewed in Appendix A. For a detailed analysis, see e.g. [22] and references therein.

Every section $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ satisfying $i \cdot \hat{\gamma} = j_1(\pi \cdot \hat{\gamma})$ will be called *admissible*. With this terminology, the admissible sections of \mathcal{A} are in 1–1 correspondence with the admissible evolutions $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$. The manifold \mathcal{A} will be referred to local fibered coordinates $t, q^1, \dots, q^n, z^1, \dots, z^r$, with transformation laws

$$\bar{t} = t, \quad \bar{q}^i = \bar{q}^i(t, q^1, \dots, q^n), \quad \bar{z}^A = \bar{z}^A(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad (2.9)$$

The imbedding $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ will be locally expressed as

$$\dot{q}^i = \psi^i(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad i = 1, \dots, n \quad (2.10)$$

with $\text{rank} \left\| \frac{\partial(\psi^1 \dots \psi^n)}{\partial(z^1 \dots z^r)} \right\| = r$. With this notation, given any section $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ described in coordinates as $q^i = q^i(t)$, $z^A = z^A(t)$, the admissibility requirement $i \cdot \hat{\gamma} = j_1(\pi \cdot \hat{\gamma})$ takes the explicit form

$$\frac{dq^i}{dt} = \psi^i(t, q^1(t), \dots, q^n(t), z^1(t), \dots, z^r(t)) \quad (2.11)$$

Eqs. (2.11) indicates that, for any admissible evolution of the system, the knowledge of the functions $z^A(t)$ determines $q^i(t)$ up to initial data.

On the other hand, in the absence of specific assumptions on the nature of the manifold \mathcal{A} , such as e.g. the existence of a canonical factorization of the form $\mathcal{Z} \times_{\mathbb{R}} \mathcal{V}_{n+1}$, \mathcal{Z} denoting an $(r+1)$ -dimensional fiber bundle over \mathbb{R} with coordinates t, z^1, \dots, z^r , the functions $z^A(t)$, in themselves, have no invariant geometrical meaning.

To pursue the idea of the z^A 's as “handles” controlling the evolution of the system, attention should rather be shifted on *sections* $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$.

Every such section will be called a *control* for the system; the composite map $i \cdot \sigma : \mathcal{V}_{n+1} \rightarrow j_1(\mathcal{V}_{n+1})$ will be called an *admissible velocity field*.

In local coordinates we have the representations

$$\sigma : z^A = z^A(t, q^1, \dots, q^n) \quad (2.12a)$$

$$i \cdot \sigma : \dot{q}^i = \psi^i(t, q^1, \dots, q^n, z^A(t, q^1, \dots, q^n)) \quad (2.12b)$$

confirming that the knowledge of σ does indeed determine the evolution of the system from any given initial event $x \in \mathcal{V}_{n+1}$ through a well posed Cauchy problem.

A section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ and a control $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$ will be said to *belong* to each other if and only if the lift $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ factors into $\hat{\gamma} = \sigma \cdot \gamma$, i.e. if and only if the jet extension $j_1(\gamma)$ coincides with the composite map $i \cdot \sigma \cdot \gamma : \mathbb{R} \rightarrow j_1(\mathcal{V}_{n+1})$.

(v) Given any admissible section $\hat{\gamma}$, let $A(\hat{\gamma}) \xrightarrow{t} \mathbb{R}$ denote the vector bundle formed by the totality of vectors along $\hat{\gamma}$ annihilating the 1-form dt . For each $t \in \mathbb{R}$, the fiber of $A(\hat{\gamma})$ over t will be denoted by $A(\hat{\gamma})|_t$.

On account of Proposition 2.1, the push-forward $i_* : T(\mathcal{A}) \rightarrow T(j_1(\mathcal{V}_{n+1}))$ gives rise to a bundle morphism

$$\begin{array}{ccc} A(\hat{\gamma}) & \xrightarrow{i_*} & j_1(V(\gamma)) \\ \pi_* \downarrow & & \downarrow \pi_* \\ V(\gamma) & \xlongequal{\quad} & V(\gamma) \end{array} \quad (2.13)$$

making $A(\hat{\gamma})$ into a subbundle of $j_1(V(\gamma))$, fibered over $V(\gamma)$.

Once again all arrows in diagram (2.13), regarded as maps between vector bundles over \mathbb{R} , have the nature of homomorphisms. The kernel of the projection $A(\hat{\gamma}) \xrightarrow{\pi_*} V(\gamma)$, clearly identical to the restriction of the vertical bundle $V(\mathcal{A})$ to the curve $\hat{\gamma}$, will be denoted by $V(\hat{\gamma})$, and will be called the *vertical bundle* along $\hat{\gamma}$.

Every fibered coordinate system t, q^i, z^A in \mathcal{A} induces coordinates t, u^i, v^A in $A(\hat{\gamma})$ according to the prescription

$$\hat{X} = u^i(\hat{X}) \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}(t(\hat{X}))} + v^A(\hat{X}) \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}(t(\hat{X}))} \quad \forall \hat{X} \in A(\hat{\gamma}) \quad (2.14)$$

In terms of these, and of the jet coordinates t, u^i, \dot{u}^i on $j_1(V(\gamma))$, the morphism (2.13) is locally described by the system

$$t = t, \quad u^i = u^i, \quad \dot{u}^i = \left(\frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} u^k + \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} v^A \quad (2.15)$$

while the vertical bundle $V(\hat{\gamma})$ coincides with the slice $u^i = 0$ in $A(\hat{\gamma})$.

(vi) For later use, let us finally examine the “constrained” counterpart of diagram (2.4). To this end we observe that the morphism (2.13) maps $V(\hat{\gamma})$ into the vertical subbundle $V(j_1(\gamma)) \subset j_1(V(\gamma))$. Composing with the morphism (2.4), and setting $\hat{\varrho} := \varrho \cdot i_*$, this gives rise to an injective homomorphism

$$\begin{array}{ccc} V(\hat{\gamma}) & \xrightarrow{\hat{\varrho}} & V(\gamma) \\ t \downarrow & & \downarrow t \\ \mathbb{R} & \xlongequal{\quad} & \mathbb{R} \end{array} \quad (2.16)$$

In coordinates, eqs. (2.5), (2.15) provide the representation

$$\hat{\varrho} \left[Y^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \right] = Y^A \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left(\frac{\partial}{\partial q^i} \right)_{\gamma} \quad (2.17)$$

2.2 Deformations

Quite generally, given a fiber bundle $M \xrightarrow{p} N$ and a section $\nu : N \rightarrow M$, a *deformation* of ν is a 1-parameter family of sections ν_ξ , $\xi \in (-\varepsilon, \varepsilon)$ depending differentiably on ξ and satisfying $\nu_0 = \nu$.

For each $y \in N$, the curve $\xi \rightarrow \nu_\xi(y)$ is called the *orbit* of the deformation ν_ξ through the point $\nu(y)$. The vector field along ν tangent to the orbits at $\xi = 0$ is called the *infinitesimal deformation* associated with ν_ξ .

Let us apply all this to the case in study.

(i) According to the stated definition, a deformation of a control $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$ is a one parameter family of controls σ_ξ depending differentiably on ξ and satisfying $\sigma_0 = \sigma$. In coordinates, preserving the representation (2.12a), we shall write

$$\sigma_\xi : \quad z^A = \zeta^A(\xi, t, q^1, \dots, q^n) \quad (2.18)$$

with $\zeta^A(0, t, q^1, \dots, q^n) = z^A(t, q^1, \dots, q^n)$. The infinitesimal deformation associated with σ_ξ is expressed locally as $\left. \frac{\partial \zeta^A}{\partial \xi} \right|_{\xi=0} \left(\frac{\partial}{\partial z^A} \right)_\sigma$. From this it is readily seen that the totality of infinitesimal deformations of a control σ coincides with the totality of vertical vector fields along σ .

(ii) When deforming an admissible section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ care must be taken of the presence of constraints. Given any such γ , a deformation γ_ξ is called *admissible* if and only if each section $\gamma_\xi : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ is admissible. In a similar way, a deformation $\hat{\gamma}_\xi$ of an admissible section $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ is called admissible if and only if all sections $\hat{\gamma}_\xi : \mathbb{R} \rightarrow \mathcal{A}$ are admissible.

As pointed out in § 2.1, the admissible sections $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ are in 1-1 correspondence with the admissible sections $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ through the relations

$$\gamma = \pi \cdot \hat{\gamma}, \quad j_1(\gamma) = i \cdot \hat{\gamma} \quad (2.19)$$

Every admissible deformation of γ may therefore be expressed as

$$\gamma_\xi = \pi \cdot \hat{\gamma}_\xi$$

$\hat{\gamma}_\xi : \mathbb{R} \rightarrow \mathcal{A}$ denoting an admissible deformation of $\hat{\gamma}$.

In coordinates, preserving the representation $\hat{\gamma} : q^i = q^i(t), z^A = z^A(t)$, the admissible deformations of $\hat{\gamma}$ are described by equations of the form

$$\hat{\gamma}_\xi : \quad q^i = \varphi^i(\xi, t), \quad z^A = \zeta^A(\xi, t) \quad (2.20)$$

subject to the conditions

$$\varphi^i(0, t) = q^i(t), \quad \zeta^A(0, t) = z^A(t) \quad (2.21a)$$

$$\frac{\partial \varphi^i}{\partial t} = \psi^i(t, \varphi^i(\xi, t), \zeta^A(\xi, t)) \quad (2.21b)$$

Setting

$$X^i(t) := \left(\frac{\partial \varphi^i}{\partial \xi} \right)_{\xi=0}, \quad \Gamma^A(t) := \left(\frac{\partial \zeta^A}{\partial \xi} \right)_{\xi=0}$$

the infinitesimal deformation tangent to $\hat{\gamma}_\xi$ is described by the vector field

$$\hat{X} = X^i(t) \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A(t) \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \quad (2.22)$$

while eq. (2.21 b) is reflected into the relation

$$\frac{dX^i}{dt} = \frac{\partial^2 \varphi^i}{\partial t \partial \xi} \Big|_{\xi=0} = \left(\frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} X^k + \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \Gamma^A \quad (2.23)$$

commonly referred to as the *variational equation*. The infinitesimal deformation tangent to the projection $\gamma_\xi = \pi \cdot \hat{\gamma}_\xi$ is similarly described by the field

$$X = \pi_* \hat{X} = \left(\frac{\partial \varphi^i}{\partial \xi} \right)_{\xi=0} \left(\frac{\partial}{\partial q^i} \right)_\gamma = X^i(t) \frac{\partial}{\partial q^i} \quad (2.24)$$

Collecting all previous results and recalling the definitions of the vector bundles $V(\gamma)$ and $A(\hat{\gamma})$ we get the following

Proposition 2.2 *Let $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ and $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ denote two admissible sections, related by eq. (2.19). Then:*

- i) the infinitesimal deformations of γ and of $\hat{\gamma}$ are respectively expressed as sections $X : \mathbb{R} \rightarrow V(\gamma)$ and $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$;*
- ii) a section $X : \mathbb{R} \rightarrow V(\gamma)$ represents an admissible infinitesimal deformation of γ if and only if its first jet extension factors through $A(\hat{\gamma})$, i.e. if and only if there exists a section $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$ satisfying $j_1(X) = i_* \hat{X}$; conversely, a section $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$ represents an admissible infinitesimal deformation of $\hat{\gamma}$ if and only if it projects into an admissible infinitesimal deformation of γ , i.e. if and only if $i_* \hat{X} = j_1(\pi_* \hat{X})$.*

The proof is entirely straightforward, and is left to the reader.

From a structural viewpoint, Proposition 2.2 establishes a complete symmetry between the roles of diagram (2.8) in the study of the admissible *evolutions* and of diagram (2.13) in the study of the admissible infinitesimal *deformations*, thus enforcing the intuitive idea that the latter context is essentially a “linearized counterpart” of the former one.

2.3 Infinitesimal controls

According to Proposition 2.2, the admissible infinitesimal deformations of an admissible section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ are in 1–1 correspondence with the sections $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$ satisfying the consistency requirement $i_* \hat{X} = j_1(\pi_* \hat{X})$.

In local coordinates, setting $\hat{X} = X^i(t) \frac{\partial}{\partial q^i} + \Gamma^A(t) \frac{\partial}{\partial z^A}$, the stated requirement is expressed by the variational equation

$$\frac{dX^i}{dt} = \frac{\partial \psi^i}{\partial q^k} X^k + \frac{\partial \psi^i}{\partial z^A} \Gamma^A \quad (2.25)$$

all coefficients being evaluated along the curve $\hat{\gamma}$.

Exactly as it happened with eq. (2.11), eq. (2.25) indicates that, for each admissible \hat{X} , the knowledge of the functions $\Gamma^A(t)$ determines the remaining $X^i(t)$ up to initial data, through the solution of a well posed Cauchy problem.

Once again, however, the drawback is that the components Γ^A , in themselves, have no invariant geometrical meaning, but obey the non-homogeneous transformation law

$$\bar{\Gamma}^A = \frac{\partial \bar{z}^A}{\partial q^i} X^i + \frac{\partial \bar{z}^A}{\partial z^B} \Gamma^B$$

under arbitrary coordinate transformation. The difficulty is overcome introducing a linearized version of the idea of *control*.

Definition 2.1 Let $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ denote an admissible evolution. Then:

- a linear section $h : V(\gamma) \rightarrow A(\hat{\gamma})$, meant as a vector bundle homomorphism satisfying $\pi_* \cdot h = id$, is called an *infinitesimal control* along γ ;
- the image $\mathcal{H}(\hat{\gamma}) := h(V(\gamma))$, viewed as a vector subbundle of $A(\hat{\gamma}) \rightarrow \mathbb{R}$, is called the *horizontal distribution* along $\hat{\gamma}$ induced by h ; every section $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$ satisfying $\hat{X}(t) \in \mathcal{H}(\hat{\gamma}) \forall t \in \mathbb{R}$ is called a *horizontal section*.

Remark 2.1 The term *infinitesimal control* is intuitively clear: given an admissible section γ , let $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$ denote any control belonging to γ , i.e. satisfying $\sigma \cdot \gamma = \hat{\gamma}$. Then, on account of the identity $\pi_* \cdot \sigma_* = (\pi \cdot \sigma)_* = id$, the tangent map $\sigma_* : T(\mathcal{V}_{n+1}) \rightarrow T(\mathcal{A})$, restricted to $V(\gamma)$, determines a linear section $\sigma_* : V(\gamma) \rightarrow A(\hat{\gamma})$. The infinitesimal controls may therefore be thought of as equivalence classes of ordinary controls having a first order contact along γ .

Given an infinitesimal control $h : V(\gamma) \rightarrow A(\hat{\gamma})$, on account of Definition 2.1 it is easily seen that the horizontal distribution $\mathcal{H}(\hat{\gamma})$ and the vertical subbundle $V(\hat{\gamma})$ split the vector bundle $A(\hat{\gamma})$ into the fibered direct sum

$$A(\hat{\gamma}) = \mathcal{H}(\hat{\gamma}) \oplus_{\mathbb{R}} V(\hat{\gamma}) \quad (2.26)$$

thus giving rise to a couple of homomorphisms $\mathcal{P}_H : A(\hat{\gamma}) \rightarrow \mathcal{H}(\hat{\gamma})$ (horizontal projection) and $\mathcal{P}_V : A(\hat{\gamma}) \rightarrow V(\hat{\gamma})$ (vertical projection), uniquely defined by the relations

$$\mathcal{P}_H = h \cdot \pi_* \quad ; \quad \mathcal{P}_V = id - \mathcal{P}_H \quad (2.27)$$

In fiber coordinates, preserving the notation (2.3a), (2.14), every infinitesimal control $h : V(\gamma) \rightarrow A(\hat{\gamma})$ is represented by a linear system of the form

$$v^A = h_i^A(t) u^i \quad (2.28)$$

In this way:

- the horizontal distribution $\mathcal{H}(\hat{\gamma})$ is locally spanned by the vector fields

$$\tilde{\partial}_i := h \left[\left(\frac{\partial}{\partial q^i} \right)_{\gamma} \right] = \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \quad (2.29)$$

- every vertical vector field $X = X^i(t) \left(\frac{\partial}{\partial q^i} \right)_{\gamma}$ along γ may be lifted to a horizontal field $h(X)$ along $\hat{\gamma}$, expressed in components as

$$h(X) = X^i(t) \tilde{\partial}_i = X^i(t) \left[\left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \right] \quad (2.30)$$

- every vector $\hat{X} = X^i \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \in A(\hat{\gamma})$ admits a unique representation of the form $\hat{X} = \mathcal{P}_H(\hat{X}) + \mathcal{P}_V(\hat{X})$, with

$$\mathcal{P}_H(\hat{X}) = X^i \tilde{\partial}_i, \quad \mathcal{P}_V(\hat{X}) = \left(\Gamma^A - X^i h_i^A \right) \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} := Y^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \quad (2.31)$$

The role of Definition 2.1 in the study of the variational equation (2.25) is further enhanced by the following

Definition 2.2 *Let h be an infinitesimal control for the (admissible) section γ . A section $X : \mathbb{R} \rightarrow V(\gamma)$ is said to be h -transported along γ if and only if its horizontal lift $h(X) : \mathbb{R} \rightarrow A(\hat{\gamma})$ is an admissible infinitesimal deformation of $\hat{\gamma}$, i.e. if and only if $i_* \cdot h(X) = j_1(X)$.*

In coordinates, setting $X = X^i(t) \left(\frac{\partial}{\partial q^i} \right)_{\gamma}$ and recalling eqs. (2.25), (2.30), the condition for h -transport is expressed by the linear system of ordinary differential equations

$$\frac{dX^i}{dt} = \left[\left(\frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} + h_k^A \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \right] X^k = X^k \tilde{\partial}_k \psi^i \quad (2.32)$$

From the latter, recalling Cauchy theorem, we conclude that the h -transported sections of $V(\gamma)$ form an n -dimensional vector space V_h , isomorphic to each fibre $V(\gamma)|_t$ through the evaluation map $X \rightarrow X(t)$. We have thus proved:

Proposition 2.3 *Every infinitesimal control $h : V(\gamma) \rightarrow A(\hat{\gamma})$ determines a trivialization of the vector bundle $V(\gamma) \xrightarrow{t} \mathbb{R}$.*

Proposition 2.3 provides an identification between sections $X : \mathbb{R} \rightarrow V(\gamma)$ and vector valued functions $X : \mathbb{R} \rightarrow V_h$, and therefore, by duality, also an identification between sections $\hat{\lambda} : \mathbb{R} \rightarrow V^*(\gamma)$ and vector valued functions $\hat{\lambda} : \mathbb{R} \rightarrow V_h^*$, thus allowing the introduction of an *absolute time derivative* $\frac{D}{Dt}$ for vertical vector fields and virtual 1-forms along γ .

The algorithm is readily implemented in components. To this end, let $\{e_{(a)}\}$, $\{e^{(a)}\}$ denote any pair of dual bases for the spaces V_h , V_h^* . By definition, each $e_{(a)}$

is a vertical vector field along γ , obeying the transport law (2.32). In coordinates, setting $e_{(a)} = e_{(a)}^i \left(\frac{\partial}{\partial q^i} \right)_\gamma$, this implies the relation

$$\frac{de_{(a)}^i}{dt} = e_{(a)}^k \tilde{\partial}_k \psi^i \quad (2.33a)$$

In a similar way, each $e^{(a)}$ is a contact 1-form along γ , expressed on the basis $\hat{\omega}^i$ as $e^{(a)} = e_i^{(a)} \hat{\omega}^i$, with $e_i^{(a)} e_{(b)}^i = \delta_b^a$.

On account of eq. (2.33a), the components $e_i^{(a)}$ obey the transport law

$$\frac{d}{dt} \left(e_i^{(a)} e_{(a)}^j \right) = 0 \quad \implies \quad \frac{de_i^{(a)}}{dt} = -e_j^{(a)} \tilde{\partial}_i \psi^j \quad (2.33b)$$

The functions

$$\tau_i{}^j := \frac{de_i^{(a)}}{dt} e_{(a)}^j = -e_i^{(a)} \frac{de_{(a)}^j}{dt} \quad (2.34a)$$

will be called the *temporal connection coefficients* associated with the infinitesimal control h in the coordinate system t, q^i . Comparison with eqs. (2.29), (2.33a,b) provides the representation

$$\tau_i{}^j = -\tilde{\partial}_i \psi^j = - \left(\frac{\partial \psi^j}{\partial q^i} \right)_{\hat{\gamma}} - h_i^A \left(\frac{\partial \psi^j}{\partial z^A} \right)_{\hat{\gamma}} \quad (2.34b)$$

Given any vertical vector field $X = X^i \left(\frac{\partial}{\partial q^i} \right)_\gamma$ along γ , the definition of the operator $\frac{D}{Dt}$ is summarized into the expression

$$\frac{DX}{Dt} = \frac{d}{dt} \left\langle X, e^{(a)} \right\rangle e_{(a)} = \frac{d}{dt} \left\langle X, e_i^{(a)} \hat{\omega}^i \right\rangle e_{(a)} = \frac{d}{dt} \left(X^i e_i^{(a)} \right) e_{(a)}^j \left(\frac{\partial}{\partial q^j} \right)_\gamma$$

written more simply as

$$\frac{DX}{Dt} = \left(\frac{dX^j}{dt} + X^i \tau_i{}^j \right) \left(\frac{\partial}{\partial q^j} \right)_\gamma \quad (2.35a)$$

with the coefficients $\tau_i{}^j$ given by eq. (2.34a). In a similar way, given any virtual 1-form $\hat{\lambda} = \lambda_i \hat{\omega}^i$, the same argument provides the evaluation

$$\frac{D\hat{\lambda}}{Dt} = \frac{d}{dt} \left\langle \hat{\lambda}, e_{(a)} \right\rangle e^{(a)} = \frac{d}{dt} \left(\lambda_i e_{(a)}^i \right) e_j^{(a)} \hat{\omega}^j = \left(\frac{d\lambda_j}{dt} - \lambda_i \tau_j{}^i \right) \hat{\omega}^j \quad (2.35b)$$

After these preliminaries, let us go back to the variational equation (2.25). By means of the projections (2.31), every section $\hat{X} = X^i \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$ splits into the sum

$$\hat{X} = \mathcal{P}_H(\hat{X}) + \mathcal{P}_V(\hat{X}) = h(X) + Y \quad (2.36)$$

with $X = \pi_*(X)$, $Y = \mathcal{P}_V(\hat{X}) = (\Gamma^A - h_i^A X^i) \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$.

On the other hand, on account of eq. (2.29), the variational equation (2.25) is mathematically equivalent to the relation

$$\frac{dX^i}{dt} - \tilde{\partial}_k(\psi^i) X^k = \left(-h_k^A X^k + \Gamma^A \right) \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}}$$

Recalling eqs. (2.34b), (2.35a), (2.36), as well as the representation (2.17) of the homomorphism $V(\hat{\gamma}) \xrightarrow{\hat{\varrho}} V(\gamma)$, the latter may be written synthetically as

$$\frac{DX}{Dt} = \hat{\varrho}(Y) = \hat{\varrho}(\mathcal{P}_V(\hat{X})) \quad (2.37a)$$

or also, setting $X = X^a e_{(a)}$, $Y = Y^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$, and expressing everything in components in the basis $e_{(a)}$

$$\frac{dX^a}{dt} = \left\langle e^{(a)}, \hat{\varrho}(Y) \right\rangle = e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A \quad (2.37b)$$

Exactly as the original equation (2.25), eq. (2.37a) points out that every infinitesimal deformation X is determined by the knowledge of a vertical vector field $Y = Y^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$ through the solution of a well posed Cauchy problem.

The advantage is that, in the newer formulation, all quantities have a precise geometrical meaning relative to the horizontal distribution $\mathcal{H}(\hat{\gamma})$ induced by the infinitesimal control h .

On the other hand, one cannot overlook the fact that, in the standard formulation of control theory, no distinguished section $h : V(\gamma) \rightarrow A(\hat{\gamma})$ is provided, and none is needed in order to formulate the results. In this respect, the infinitesimal control h plays the role of a *gauge field*, useful for covariance purposes, but un-affecting the evaluation of the extremals. Accordingly, in the subsequent analysis we shall employ h as a user-defined object, eventually checking the invariance of the results under arbitrary changes $h \rightarrow h'$.

2.4 Corners

To complete our geometrical setup we have still to consider the fact that, in general, control theory does not deal with sections in the ordinary sense, but with *piecewise differentiable evolutions*, defined on *closed* intervals. To account for this aspect, we adopt the following standard terminology:

- an admissible closed arc $(\gamma, [a, b])$ in \mathcal{V}_{n+1} is the restriction to a closed interval $[a, b]$ of an admissible section $\gamma : (c, d) \rightarrow \mathcal{V}_{n+1}$ defined on some open interval $(c, d) \supset [a, b]$;
- a piecewise differentiable evolution of the system in the interval $[t_0, t_1]$ is a finite collection

$$(\gamma, [t_0, t_1]) := \left\{ (\gamma^{(s)}, [a_{s-1}, a_s]), s = 1, \dots, N, t_0 = a_0 < a_1 < \dots < a_N = t_1 \right\}$$

of admissible closed arcs satisfying the matching conditions

$$\gamma^{(s)}(a_s) = \gamma^{(s+1)}(a_s) \quad \forall s = 1, \dots, N-1 \quad (2.38)$$

Due to eq. (2.38), the image $\gamma(t)$ is well defined and continuous for all $t_0 \leq t \leq t_1$, thus allowing to regard the map $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ as a section in a broad sense. The points $\gamma(t_0), \gamma(t_1)$ are called the endpoints of γ . The points $x_s := \gamma(a_s)$, $s = 1, \dots, N-1$ are called the *corners* of γ .

Consistently with the stated definitions, the lift of an admissible closed arc $(\gamma, [a, b])$ is the restriction to $[a, b]$ of the lift $\hat{\gamma} : (c, d) \rightarrow \mathcal{A}$, while the lift $\hat{\gamma}$ of a piecewise differentiable evolution $\{(\gamma^{(s)}, [a_{s-1}, a_s])\}$ is the family of lifts $\hat{\gamma}^{(s)}$, each restricted to the interval $[a_{s-1}, a_s]$.

With this definition, the image $\hat{\gamma}(t)$ is well defined for all $t \neq a_1, \dots, a_{N-1}$, thus allowing to regard $\hat{\gamma} : [t_0, t_1] \rightarrow \mathcal{A}$ as a (generally discontinuous) section of the velocity space. In particular, due to the fact that the map $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ is an imbedding of \mathcal{A} into an *affine* bundle over \mathcal{V}_{n+1} , each difference

$$[\hat{\gamma}]_{a_s} = i(\hat{\gamma}^{(s+1)}(a_s)) - i(\hat{\gamma}^{(s)}(a_s)), \quad s = 1, \dots, N-1$$

identifies a vertical vector in $T_{x_s}(\mathcal{V}_{n+1})$, henceforth called the *jump* of $\hat{\gamma}$ at the corner x_s .

In local coordinates, setting $q^i(\gamma^{(s)}(t)) := q_{(s)}^i(t)$, eqs. (2.11), (2.38) provide the representation

$$[\hat{\gamma}]_{a_s} = \left(\left(\frac{dq_{(s+1)}^i}{dt} \right)_{a_s} - \left(\frac{dq_{(s)}^i}{dt} \right)_{a_s} \right) \left(\frac{\partial}{\partial q^i} \right)_{x_s} = [\psi^i(\hat{\gamma})]_{a_s} \left(\frac{\partial}{\partial q^i} \right)_{x_s} \quad (2.39)$$

with $[\psi^i(\hat{\gamma})]_{a_s} := \psi^i(\hat{\gamma}^{(s+1)}(a_s)) - \psi^i(\hat{\gamma}^{(s)}(a_s))$ denoting the jump of the function $\psi^i(\hat{\gamma}(t))$ at $t = a_s$.

Going on with the generalization process, an admissible deformation of an admissible closed arc $(\gamma, [a, b])$ is a 1-parameter family $(\gamma_\xi, [a(\xi), b(\xi)])$, $|\xi| < \varepsilon$ of admissible closed arcs depending differentiably on ξ and satisfying the condition $(\gamma_0, [a(0), b(0)]) = (\gamma, [a, b])$. Notice that the definition explicitly includes possible variations of the reference intervals $[a(\xi), b(\xi)]$.

In a similar way, an admissible deformation of a piecewise differentiable evolution $(\gamma, [t_0, t_1])$ is a collection $\{(\gamma_\xi^{(s)}, [a_{s-1}(\xi), a_s(\xi)])\}$ of deformations of the various arcs, satisfying the matching conditions

$$\gamma_\xi^{(s)}(a_s(\xi)) = \gamma_\xi^{(s+1)}(a_s(\xi)) \quad \forall |\xi| < \varepsilon, \quad s = 1, \dots, N-1 \quad (2.40)$$

Under the stated circumstances, the lifts $\hat{\gamma}_\xi$ and $\hat{\gamma}_\xi^{(s)}$, respectively restricted to the intervals $[a(\xi), b(\xi)]$ and $[a_{s-1}(\xi), a_s(\xi)]$ are easily recognized to provide deformations for the lifts $\hat{\gamma} : [a, b] \rightarrow \mathcal{A}$ and $\hat{\gamma}^{(s)} : [a_{s-1}, a_s] \rightarrow \mathcal{A}$.

Unless otherwise stated, we shall only consider deformations leaving the interval $[t_0, t_1]$ fixed, i.e. satisfying the conditions $a_0(\xi) \equiv t_0$, $a_N(\xi) \equiv t_1$. No restriction will be posed on the functions $a_s(\xi)$, $s = 1, \dots, N-1$.

Each curve $x_s(\xi) := \gamma_\xi(a_s(\xi))$ will be called the *orbit* of the corner x_s under the given deformation.

In local coordinates, setting $q^i(\gamma_\xi^{(s)}(t)) = \varphi_{(s)}^i(\xi, t)$, the matching conditions (2.40) read

$$\varphi_{(s)}^i(\xi, a_s(\xi)) = \varphi_{(s+1)}^i(\xi, a_s(\xi)) \quad (2.41)$$

while the representation of the orbit $x_s(\xi)$ takes the form

$$x_s(\xi) : \quad t = a_s(\xi), \quad q^i = \varphi_{(s)}^i(\xi, a_s(\xi)) \quad (2.42)$$

The previous arguments are naturally reflected into the definition of the infinitesimal deformations. Thus, an admissible infinitesimal deformation of an admissible closed arc $(\gamma, [a, b])$ is a triple (α, X, β) , where X is the restriction to $[a, b]$ of an admissible infinitesimal deformation of $\gamma : (c, d) \rightarrow \mathcal{V}_{n+1}$, while α, β are the derivatives

$$\alpha = \left. \frac{da}{d\xi} \right|_{\xi=0}, \quad \beta = \left. \frac{db}{d\xi} \right|_{\xi=0} \quad (2.43)$$

expressing the *speed of variation* of the interval $[a(\xi), b(\xi)]$ at $\xi = 0$.

In a similar way, an admissible infinitesimal deformation of a piecewise differentiable evolution $(\gamma, [t_0, t_1])$ is a collection $\{\dots \alpha_{s-1}, X_{(s)}, \alpha_s \dots\}$ of admissible infinitesimal deformations of each single closed arc, with $\alpha_s = \left. \frac{da_s}{d\xi} \right|_{\xi=0}$, and, in particular, with $\alpha_0 = \alpha_N = 0$ whenever the interval $[t_0, t_1]$ is held fixed.

Let us analyse the situation in detail. To start with, we notice that the quantities $\alpha_s, X_{(s)}$ are not independent: eqs. (2.41) imply in fact the identities

$$\frac{\partial \varphi_{(s)}^i}{\partial \xi} + \frac{\partial \varphi_{(s)}^i}{\partial t} \frac{da_s}{d\xi} = \frac{\partial \varphi_{(s+1)}^i}{\partial \xi} + \frac{\partial \varphi_{(s+1)}^i}{\partial t} \frac{da_s}{d\xi}$$

From these, evaluating everything at $\xi = 0$ and recalling the relation between finite deformations and infinitesimal ones, we get the *jump relations*

$$\left(X_{(s+1)}^i - X_{(s)}^i \right)_{a_s} = -\alpha_s \left(\frac{dq_{(s+1)}^i}{dt} - \frac{dq_{(s)}^i}{dt} \right)_{a_s} = -\alpha_s \left[\psi^i(\hat{\gamma}) \right]_{a_s} \quad (2.44)$$

Moreover, the admissibility of each single infinitesimal deformation $X_{(s)}$ requires the existence of a corresponding lift $\hat{X}_{(s)} = X_{(s)}^i \left(\frac{\partial}{\partial q^i} \right)_{\hat{\gamma}^{(s)}} + \Gamma_{(s)}^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}^{(s)}}$ satisfying the variational equation (2.23).

Both aspects are conveniently accounted for assigning to each $\gamma^{(s)}$ an (arbitrarily chosen) infinitesimal control $h_{(s)} : V(\gamma^{(s)}) \rightarrow A(\hat{\gamma}^{(s)})$. In this way, proceeding as in § 2.3 and denoting by $\left(\frac{D}{Dt} \right)_{\gamma^{(s)}}$ the absolute time derivative along $\gamma^{(s)}$ induced by $h_{(s)}$ we get the following

Proposition 2.4 *Every admissible infinitesimal deformation of an admissible evolution $(\gamma, [t_0, t_1])$ over a fixed interval $[t_0, t_1]$ is determined, up to initial data, by*

a collection of vertical vector fields $\{Y_{(s)} = Y_{(s)}^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}^{(s)}}\}$, $s = 1, \dots, N$ and by $N - 1$ real numbers $\alpha_1, \dots, \alpha_{N-1}$ through the covariant variational equations

$$\left(\frac{DX_{(s)}}{Dt}\right)_{\gamma^{(s)}} = \hat{\varrho}(Y_{(s)}) = Y_{(s)}^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}} \quad s = 1, \dots, N \quad (2.45)$$

completed with the jump conditions (2.44). The lift of the deformation is described by the family of vector fields

$$\hat{X}_{(s)} = h_{(s)}(X_{(s)}) + Y_{(s)}, \quad s = 1, \dots, N \quad (2.46)$$

The proof is entirely straightforward, and is left to the reader. Introducing n piecewise differentiable vector fields $\tilde{\partial}_1, \dots, \tilde{\partial}_n$ along $\hat{\gamma}$ according to the prescription

$$\tilde{\partial}_i(t) = h_{(s)} \left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}(t)} \quad \forall t \in (a_{s-1}, a_s), \quad s = 1, \dots, N$$

eq. (2.46) takes the explicit form

$$\hat{X}_{(s)} = h_{(s)} \left(X_{(s)}^i \left(\frac{\partial}{\partial q^i}\right)_{\gamma}\right) + Y_{(s)} = X_{(s)}^i \tilde{\partial}_i + Y_{(s)}^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \quad (2.47)$$

on each open arc $\hat{\gamma}^{(s)} : (a_{s-1}, a_s) \rightarrow \mathcal{A}$.

To discuss the implications of eq. (2.45) we resume the notation $V(\gamma)$ for the totality of vertical vectors along γ . Notice that this makes perfectly good sense also at the corners $\gamma(a_s)$.

We then define a transport law in $V(\gamma)$, henceforth called h -transport, gluing $h_{(s)}$ -transport along each arc $(\gamma^{(s)}, [a_{s-1}, a_s])$ and continuity at the corners, i.e. continuity of the components at $t = a_s$.

In view of Proposition 2.3, the h -transported fields form an n -dimensional vector space V_h , isomorphic to each fibre $V(\gamma)|_t$.

This provides a canonical identification of $V(\gamma)$ with the cartesian product $[t_0, t_1] \times V_h$, thus allowing to regard every section $X : [t_0, t_1] \rightarrow V(\gamma)$ as a vector valued function $X : [t_0, t_1] \rightarrow V_h$.

Exactly as in § 2.3, the situation is formalized referring V_h to a basis $\{e_{(a)}\}$ related to the basis $\left(\frac{\partial}{\partial q^i}\right)_{\gamma}$ by the transformation

$$\left(\frac{\partial}{\partial q^i}\right)_{\gamma} = e_i^{(a)}(t) e_{(a)}, \quad e_{(a)} = e_{(a)}^i(t) \left(\frac{\partial}{\partial q^i}\right)_{\gamma} \quad (2.48)$$

Given any admissible infinitesimal deformation $\{(X_{(s)}, [a_{s-1}, a_s])\}$, we now glue all sections $X_{(s)} : [a_{s-1}, a_s] \rightarrow V(\gamma^{(s)})$ into a single, piecewise differentiable function $X : [t_0, t_1] \rightarrow V_h$, with jump discontinuities at $t = a_s$ expressed in components by eq. (2.44). For each $s = 1, \dots, N$ this provides the representation

$$X_{(s)} = X^a(t) e_{(a)}, \quad \left(\frac{DX_{(s)}}{Dt}\right)_{\gamma^{(s)}} = \frac{dX^a}{dt} e_{(a)} \quad \forall t \in (a_{s-1}, a_s) \quad (2.49)$$

In a similar way, we collect all fields $Y_{(s)}$ into a single object Y , henceforth conventionally called a vertical vector field along $\hat{\gamma}$.

By abuse of language, we also denote by $Y = Y^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$ the vector field along the open arcs of $\hat{\gamma}$ defined by the prescription.

$$Y^A(t) = Y_{(s)}^A(t) \quad a_{s-1} < t < a_s, \quad s = 1, \dots, N \quad (2.50)$$

In this way, the covariant variational equation (2.45) takes the form

$$\frac{dX^a}{dt} = Y^A e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \quad \forall t \neq a_s \quad (2.51a)$$

completed with the jump conditions

$$[X^a]_{a_s} = [X^i]_{a_s} e_i^{(a)}(a_s) = -\alpha_s e_i^{(a)}(a_s) \left[\psi^i(\hat{\gamma}) \right]_{a_s} \quad s = 1, \dots, N-1 \quad (2.51b)$$

2.5 The variational setup

A deeper insight into the algorithm discussed in § 2.4 is gained denoting by \mathfrak{W} the infinite dimensional vector space formed by the totality of vertical vector fields $Y = \{Y_{(s)}, s = 1, \dots, N\}$ along $\hat{\gamma}$, and setting $\mathfrak{W} := \mathfrak{W} \oplus \mathbb{R}^{N-1}$. On account of eqs. (2.51a,b), every admissible infinitesimal deformation of γ is then determined, up to initial data, by an element $(Y, \alpha_1, \dots, \alpha_{N-1}) \in \mathfrak{W}$.

Let us now focus on the fact that, in the development of control theory, one is mainly interested in infinitesimal deformations $X : [t_0, t_1] \rightarrow V(\gamma)$ vanishing at the endpoints. Setting $X(t_0) = 0$, eqs. (2.51a,b) provide the evaluation

$$X(t) = \left(\int_{t_0}^t Y^A e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} dt - \sum_{a_s < t} \alpha_s e_i^{(a)}(a_s) \left[\psi^i(\hat{\gamma}) \right]_{a_s} \right) e_{(a)} \quad (2.52)$$

The vanishing of both $X(t_0)$ and $X(t_1)$ is therefore expressed by the condition

$$\left(\int_{t_0}^{t_1} Y^A e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} dt - \sum_{s=1}^{N-1} \alpha_s e_i^{(a)}(a_s) \left[\psi^i(\hat{\gamma}) \right]_{a_s} \right) e_{(a)} = 0 \quad (2.53)$$

The left hand side of eq. (2.53) defines a linear map $\Upsilon : \mathfrak{W} \rightarrow V_h$. According to the previous discussion, the kernel $\ker(\Upsilon) \subset \mathfrak{W}$ is isomorphic to the vector space of the admissible infinitesimal deformations vanishing at the end points of γ .

Equally important is the nature of the inclusion $\Upsilon(\mathfrak{W}) \subset V_h$. Depending on the latter, the evolutions of the system will be classified into *normal*, when $\Upsilon(\mathfrak{W}) = V_h$ and *abnormal*, when $\Upsilon(\mathfrak{W}) \subsetneq V_h$. As we shall see, when applied to the extremals of an action functional, this terminology agrees with the current one (see, among others, [18, 20] and references therein). The dimension of the annihilator $(\Upsilon(\mathfrak{W}))^0 \subset V_h^*$ will be called the *abnormality index* of γ .

In this connection, a useful characterization is provided by the following

Proposition 2.5 *The annihilator $(\Upsilon(\mathfrak{W}))^0 \subset V_h^*$ coincides with the totality of h -transported virtual 1-forms $\hat{\rho} = \rho_i \hat{\omega}^i$ satisfying the conditions*

$$\rho_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} = 0 \quad A = 1, \dots, r \quad (2.54a)$$

$$\rho_i(a_s) [\psi^i(\hat{\gamma})]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (2.54b)$$

Proof. In view of eq. (2.53), the subspace $(\Upsilon(\mathfrak{W}))^0 \subset V_h^*$ consists of the totality of elements $\hat{\rho} = \rho_a e^{(a)} = \rho_a e_i^{(a)} \hat{\omega}^i$ satisfying the relation

$$\rho_a \left(\int_{t_0}^{t_1} Y^A e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} dt - \sum_{s=1}^{N-1} \alpha_s e_i^{(a)}(a_s) [\psi^i(\hat{\gamma})]_{a_s} \right) = 0$$

$\forall (Y, \alpha_1, \dots, \alpha_{N-1}) \in \mathfrak{W}$, clearly equivalent to eqs. (2.54a,b). \square

On account of eqs. (2.34b), (2.35b), the condition of h -transport of $\hat{\rho}$ along each arc $\gamma^{(s)}$ is expressed in coordinates as

$$\frac{d\rho_i}{dt} + \rho_k \left(\frac{\partial \psi^k}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A \rho_k \left(\frac{\partial \psi^k}{\partial z^A} \right)_{\hat{\gamma}} = 0 \quad (2.55)$$

the cancellation arising from the requirement (2.54a).

The content of Proposition 2.5 is therefore independent of the choice of the infinitesimal controls $h_{(s)} : V(\gamma^{(s)}) \rightarrow A(\hat{\gamma}^{(s)})$.

Remark 2.2 *According to Proposition 2.5, the abnormality index of a piecewise differentiable section γ cannot exceed the abnormality index of each single arc $\gamma^{(s)}$. Thus, for example, if one of the arcs is normal, γ is necessarily normal. More generally, due to the additional restrictions posed by eqs. (2.54b) and by the continuity requirements $[\hat{\rho}]_{a_s} = 0$, an evolution may happen to be normal even if all its arcs $\gamma^{(s)}$ are abnormal. Typical examples are:*

- $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$, referred to coordinates t, x, y . Constraint: $\dot{x}^2 + \dot{y}^2 = v^2$. Imbedding $\mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v \cos z, \dot{y} = v \sin z$. Piecewise differentiable evolution γ consisting of two arcs:

$$\begin{aligned} \gamma^{(1)} : \quad & x = 0, \quad y = vt \quad t_0 \leq t \leq 0 \\ \gamma^{(2)} : \quad & x = vt, \quad y = 0 \quad 0 \leq t \leq t_1 \end{aligned}$$

Eq. (2.54a) admits h -transported solutions $\hat{\rho}^{(1)} = \alpha \hat{\omega}^1$ along $\gamma^{(1)}$ and $\hat{\rho}^{(2)} = \beta \hat{\omega}^2$ along $\gamma^{(2)}$, $\forall \alpha, \beta \in \mathbb{R}$. Both arcs are therefore abnormal. Notwithstanding, γ is normal, since no pair $\hat{\rho}^{(1)}, \hat{\rho}^{(2)}$ matches into a continuous non-null virtual 1-form along γ .

- $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$. Coordinates t, x, y . Constraint: $v^3 \dot{x} = (\dot{y}^2 - a^2 t^2)^2$. Imbedding

$\mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v^{-3}(z^2 - a^2 t^2)^2$, $\dot{y} = z$. Piecewise differentiable evolution γ consisting of two arcs:

$$\begin{aligned} \gamma^{(1)} : \quad x &= 0, & y &= \frac{1}{2} a(t^2 - t^{*2}) & t_0 \leq t \leq t^* \\ \gamma^{(2)} : \quad x &= \frac{a^4}{5v^3} (t^5 - t^{*5}), & y &= 0 & t^* \leq t \leq t_1 \end{aligned}$$

($t^* \neq 0$). Eq. (2.54a) admits h -transported solutions of the form $\hat{\rho} = \alpha \hat{\omega}^1$ along the whole of γ . Both arcs $\gamma^{(1)}$, $\gamma^{(2)}$ are therefore abnormal. Notwithstanding, γ is normal, since no solution satisfies condition (2.54b).

As a concluding remark, let us finally observe that, although geometrically significant, the arguments discussed so far provide only a partial picture of the situation. In a variational context, in fact, what really matters is not the totality of admissible infinitesimal deformations vanishing at the end points — here identified with the kernel of the map $\Upsilon : \mathfrak{W} \rightarrow V_h$ — but the (possibly smaller) subfamily \mathfrak{X} of infinitesimal deformations tangent to admissible *finite* deformations with fixed end points.

The linear span of \mathfrak{X} , henceforth denoted by $\Delta(\gamma)$, will be called the *variational space* of γ . The evolutions of the system will be classified into *ordinary*, when $\Delta(\gamma) = \ker(\Upsilon)$ and *exceptional*, when $\Delta(\gamma) \subsetneq \ker(\Upsilon)$. A hierarchy between the various typologies is provided by the following

Proposition 2.6 *The normal evolutions form a subset of the ordinary ones.*

The result is proved in Appendix B. In this connection, see also [24].

3 Calculus of variations

3.1 Extremals

Let us now come to the central problem of control theory. Let $L \in \mathfrak{F}(\mathcal{A})$ denote a differentiable function on the velocity space \mathcal{A} , henceforth called the *Lagrangian*.

Also, let $(\gamma, [t_0, t_1])$ (γ for short) denote an admissible piecewise differentiable evolution of the system, defined on a closed interval $[t_0, t_1] \subset \mathbb{R}$.

Indicating by $\hat{\gamma}$ the lift of γ , define the *action functional*

$$I[\gamma] := \int_{\hat{\gamma}} L dt := \sum_{s=1}^N \int_{a_{s-1}}^{a_s} (\hat{\gamma}^{(s)})^*(L) dt \quad (3.1)$$

Definition 3.1 *An admissible evolution γ is called an extremal for the functional (3.1) if and only if, for all admissible deformations $\gamma_\xi = \{(\gamma_\xi^{(s)}, [a_{s-1}(\xi), a_s(\xi)])\}$ with fixed endpoints, the function*

$$I(\xi) := \int_{\hat{\gamma}_\xi} L dt = \sum_{s=1}^N \int_{a_{s-1}(\xi)}^{a_s(\xi)} (\hat{\gamma}_\xi^{(s)})^*(L) dt$$

is stationary at $\xi = 0$.

The content of Definition 3.1 is formalized denoting by $\hat{X}_{(s)}$ the infinitesimal deformation associated with each single $\hat{\gamma}_\xi^{(s)}$. Recalling eq. (2.47) as well as the definition $\alpha_s = \frac{da_s}{d\xi}|_{\xi=0}$, this provides the evaluation

$$\begin{aligned} \frac{dI}{d\xi}\Big|_{\xi=0} &= \sum_{s=1}^N \left[\frac{d}{d\xi} \int_{a_{s-1}(\xi)}^{a_s(\xi)} L(\hat{\gamma}_\xi^{(s)}) dt \right]_{\xi=0} = \\ &= \sum_{s=1}^N \left\{ \int_{a_{s-1}}^{a_s} \hat{X}_{(s)}(L) dt + \left[\alpha_s L(\hat{\gamma}^{(s)}(a_s)) - \alpha_{s-1} L(\hat{\gamma}^{(s)}(a_{s-1})) \right] \right\} \end{aligned} \quad (3.2a)$$

On account of the assumption $\alpha_0 = \alpha_N = 0$, denoting by

$$[L(\hat{\gamma})]_{a_s} := [L(\hat{\gamma}^{(s+1)}(a_s)) - L(\hat{\gamma}^{(s)}(a_s))]$$

the jump of the function $L(\hat{\gamma}(t))$ at $t = a_s$, eq. (3.2a) may be concisely written as

$$\frac{dI}{d\xi}\Big|_{\xi=0} = \sum_{s=1}^N \int_{a_{s-1}}^{a_s} \left(X_{(s)}^i \tilde{\partial}_i(L) + Y_{(s)}^A \frac{\partial L}{\partial z^A} \right) dt - \sum_{s=1}^{N-1} \alpha_s [L(\hat{\gamma})]_{a_s} \quad (3.2b)$$

Eq. (3.2b) is further elaborated introducing N virtual 1-form $\hat{\lambda}^{(s)} = p_i^{(s)}(t) \hat{\omega}^i$ (one for each arc $\gamma^{(s)}$) satisfying the transport law

$$\left(\frac{D\hat{\lambda}^{(s)}}{Dt} \right)_{\gamma^{(s)}} = \left(\tilde{\partial}_i L \right)_{\hat{\gamma}^{(s)}} \hat{\omega}^i \quad (3.3a)$$

as well as the matching conditions

$$\hat{\lambda}^{(s)}|_{a_s} = \hat{\lambda}^{(s+1)}|_{a_s} \quad s = 1, \dots, N-1 \quad (3.3b)$$

Once again, for notational convenience, we collect all $\hat{\lambda}^{(s)}$ into a continuous, piecewise differentiable section $\hat{\lambda} : [t_0, t_1] \rightarrow V^*(\gamma)$ according to the prescription

$$\hat{\lambda}(t) = \hat{\lambda}^{(s)}(t) \quad \forall t \in [a_{s-1}, a_s] \quad (3.4)$$

On account of eqs. (3.3a,b), $\hat{\lambda}$ is then uniquely determined by L , up to initial data at $t = t_0$.

Taking the covariant variational equation (2.45) as well as the duality relations $\left\langle \left(\frac{\partial}{\partial q^i} \right)_{\gamma(s)}, \hat{\omega}^k \right\rangle = \delta_i^k$ into account, by eq. (3.3a) we get the expression

$$\begin{aligned} X_{(s)}^i \tilde{\partial}_i L &= \left\langle X_{(s)}, \left(\frac{D\hat{\lambda}^{(s)}}{Dt} \right)_{\gamma(s)} \right\rangle = \frac{d}{dt} \left\langle X_{(s)}, \hat{\lambda}^{(s)} \right\rangle - \left\langle \left(\frac{DX_{(s)}}{Dt} \right)_{\gamma(s)}, \hat{\lambda}^{(s)} \right\rangle = \\ &= \frac{d}{dt} \left(X_{(s)}^i p_i^{(s)} \right) - p_i^{(s)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}(s)} Y_{(s)}^A \end{aligned}$$

whence also

$$\int_{a_{s-1}}^{a_s} X_{(s)}^i \tilde{\partial}_i(L) dt = \left[X_{(s)}^i p_i^{(s)} \right]_{a_{s-1}}^{a_s} - \int_{a_{s-1}}^{a_s} p_i^{(s)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}(s)} Y_{(s)}^A dt$$

Summing over s , restoring the notations (2.50), (3.4) and recalling eqs. (2.44), (3.3b) as well as the conditions $X(t_0) = X(t_1) = 0$, this implies the relation

$$\sum_{s=1}^N \int_{a_{s-1}}^{a_s} X_{(s)}^i \tilde{\partial}_i(L) dt = - \int_{t_0}^{t_1} p_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A dt + \sum_{s=1}^{N-1} \alpha_s \left[\psi^i(\hat{\gamma}) \right]_{a_s} p_i(a_s)$$

In this way, omitting all unnecessary subscripts, eq. (3.2b) gets the final form

$$\left. \frac{dI}{d\xi} \right|_{\xi=0} = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial z^A} - p_i \frac{\partial \psi^i}{\partial z^A} \right) Y^A dt + \sum_{s=1}^{N-1} \alpha_s \left[p_i(t) \psi^i(\hat{\gamma}) - L(\hat{\gamma}) \right]_{a_s} \quad (3.5)$$

In the algebraic environment introduced in § 2.5, the previous discussion is naturally formalized regarding the right hand side of eq. (3.5) as a linear functional $dI_\gamma : \mathfrak{W} \rightarrow \mathbb{R}$ on the vector space $\mathfrak{W} = \mathfrak{V} \oplus \mathbb{R}^{N-1}$. A necessary and sufficient condition for γ to be an extremal for the functional (3.1) is then the vanishing of dI_γ on the subset $\mathfrak{X} \subset \mathfrak{W}$ formed by the totality of elements $Y, \alpha_1, \dots, \alpha_{N-1}$ arising from finite deformations with fixed end points. By linearity, the previous condition is mathematically equivalent to the requirement

$$\Delta(\gamma) \subset \ker(dI_\gamma) \quad (3.6)$$

with $\Delta(\gamma) = \text{Span}(\mathfrak{X}) \subseteq \ker(\Upsilon)$ denoting the variational space of γ .

As we shall see, eq. (3.6) provides an algorithm for the determination of *all* the extremals of the functional (3.1) within the class of *ordinary* evolutions.

The exceptional case is considerably more complicated, due to the lack of an explicit characterization of the space $\Delta(\gamma)$ in terms of the local properties of the section γ . In this respect, the simplest procedure and, quite often, the only available one, is checking eq. (3.6) separately on each exceptional evolution.

In what follows we shall adopt an intermediate strategy, namely, rather than dealing with eq. (3.6) we shall discuss the implications of the stronger requirement

$$\ker(\Upsilon) \subset \ker(dI_\gamma) \quad (3.7a)$$

According to the classification introduced in § 2.5, the latter is *necessary and sufficient* for an *ordinary* evolution γ to be an extremal of the functional (3.1), but merely *sufficient* for an *exceptional* evolution to be an extremal.

By elementary algebra, the requirement (3.7a) is equivalent to the existence of a (possibly non-unique) linear functional $K : V_h \rightarrow \mathbb{R}$ satisfying the relation

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\Upsilon} & V_h \\ & \searrow dI_\gamma & \downarrow K \\ & & \mathbb{R} \end{array} \quad (3.7b)$$

Setting $K = K_a e^{(a)}$, and recalling eqs. (2.53), (3.5), the requirement (3.7b) is expressed in components as

$$\begin{aligned} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial z^A} - p_i \frac{\partial \psi^i}{\partial z^A} \right) Y^A dt + \sum_{s=1}^{N-1} \alpha_s \left[p_i(t) \psi^i(\hat{\gamma}) - L(\hat{\gamma}) \right]_{a_s} = \\ K_a \left(\int_{t_0}^{t_1} Y^A e_i^{(a)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} dt - \sum_{s=1}^{N-1} \alpha_s e_i^{(a)}(a_s) \left[\psi^i(\hat{\gamma}) \right]_{a_s} \right) \end{aligned}$$

By the arbitrariness of $Y, \alpha_1, \dots, \alpha_{N-1}$, the latter condition splits into the system

$$\frac{\partial L}{\partial z^A} - \left(p_i + K_a e_i^{(a)} \right) \frac{\partial \psi^i}{\partial z^A} = 0 \quad A = 1, \dots, r \quad (3.8a)$$

$$\left[\left(p_i + K_a e_i^{(a)} \right) \psi^i(\hat{\gamma}) - L(\hat{\gamma}) \right]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (3.8b)$$

Collecting all results, and recalling Propositions 2.5, 2.6 we conclude

Theorem 3.1 *Given an admissible evolution γ , let $\wp(\gamma)$ denote the totality of piecewise differentiable virtual 1-forms $\hat{\lambda} = p_i(t) \hat{\omega}^i$ along γ satisfying eqs. (3.3a,b), (3.4) as well as the finite relations*

$$p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial L}{\partial z^A} \quad A = 1, \dots, r \quad (3.9a)$$

and the matching conditions

$$\left[p_i \psi^i(\hat{\gamma}) - L(\hat{\gamma}) \right]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (3.9b)$$

Then:

- a) the condition $\wp(\gamma) \neq \emptyset$ is sufficient for γ to be an extremal for the functional (3.1);

b) if γ is an ordinary evolution, the same condition is also necessary for γ to be an extremal;

c) γ is a normal extremal, namely an extremal belonging to the class of normal evolutions, if and only if the set $\wp(\gamma)$ consists of a single element.

Proof. In view of eqs. (3.5), (3.9a,b), the ansatz $\hat{\lambda} \in \wp(\gamma)$, if allowed, implies $\frac{dI}{d\xi}|_{\xi=0} = 0$ for all admissible infinitesimal deformations vanishing at the end points of γ . Assertion a) is then a direct consequence of Definition 3.1.

In particular, according to our previous discussion, if γ is an ordinary extremal, for any continuous virtual 1-form $\hat{\lambda} = p_i \hat{\omega}^i$ obeying the transport law (3.3a) there exists at least one h -transported 1-form $K = K_a e^{(a)}$ satisfying eq. (3.8a,b). The sum $\hat{\lambda} + K = (p_i + K_a e_i^{(a)}) \hat{\omega}^i$ is then automatically in the class $\wp(\gamma)$, thus proving assertion b).

Finally, as pointed out in § 2.4, the normal evolutions form a subclass of the ordinary ones, uniquely characterized by the requirement $(\Upsilon(\mathfrak{W}))^0 = \{0\}$. Therefore, according to assertion b), a normal evolution γ is an extremal if and only if the class $\wp(\gamma)$ is nonempty. Moreover, by eqs. (3.3a), (3.8a), if $\hat{\lambda}, \hat{\lambda}'$ is any pair of elements in the class $\wp(\gamma)$, the difference $\hat{\lambda} - \hat{\lambda}'$ is automatically an h -transported 1-form satisfying eqs. (2.54a,b). By Proposition 2.5 this implies $\hat{\lambda} - \hat{\lambda}' \in (\Upsilon(\mathfrak{W}))^0 \Rightarrow \hat{\lambda} = \hat{\lambda}'$, thus establishing assertion c). \square

In view of eqs. (2.34b), (2.35b), for any $\hat{\lambda} \in \wp(\gamma)$ the transport law (3.3a) simplifies to

$$\frac{dp_i}{dt} + p_k \left(\frac{\partial \psi^k}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A p_k \left(\frac{\partial \psi^k}{\partial z^A} \right)_{\hat{\gamma}} = \left(\frac{\partial L}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A \left(\frac{\partial L}{\partial z^A} \right)_{\hat{\gamma}}$$

the cancellation being due to eq. (3.9a). Exactly as it happened with Proposition 2.5, all assertions of Theorem 3.1 have therefore an intrinsic meaning, irrespective of the choice of the infinitesimal controls $h_{(s)} : V(\gamma^{(s)}) \rightarrow A(\hat{\gamma}^{(s)})$.

The previous arguments provide an algorithm for the determination of the *ordinary* extremals of the functional (3.1), relying on $2n + r$ equations

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A) \quad (3.10a)$$

$$\frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i} p_k = \frac{\partial L}{\partial q^i} \quad (3.10b)$$

$$p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial L}{\partial z^A} \quad (3.10c)$$

for the unknowns $q^i(t), p_i(t), z^A(t)$, completed with the continuity requirements

$$[q^i]_{a_s} = [p_i]_{a_s} = [p_i \psi^i - L]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (3.11)$$

As already pointed out, all equations are independent of the choice of the infinitesimal controls, and involve only the “true” data of the problem, namely the Lagrangian L and the constraint equations (2.10). In particular:

- the algorithm (3.10 a,b,c), (3.11) is invariant under arbitrary transformations of the form

$$L \rightarrow L + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k} \psi^k, \quad p_i(t) \rightarrow p_i(t) + \left(\frac{\partial f}{\partial q^i} \right)_{\gamma(t)} \quad (3.12)$$

$f(t, q^1, \dots, q^n)$ being any differentiable function over \mathcal{V}_{n+1} ;

- the last pair of equations (3.11) extend to the ordinary evolutions the well known *Erdmann–Weierstrass corner conditions* of holonomic variational calculus [9, 19].

3.2 Lagrange multipliers

For completeness, we discuss the relation between Theorem 3.1 and the traditional approach to constrained variational calculus based on the so called *Lagrange multipliers* method. To this end, we consider a situation in which:

- the behaviour of the system is determined by a “free” lagrangian \mathcal{L} , viewed as a function on $j_1(\mathcal{V}_{n+1})$, with local expression $\mathcal{L} = \mathcal{L}(t, q^i, \dot{q}^i)$;
- the presence of the constraints restricts the mobility of the system to a submanifold $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$, implicitly represented as

$$g^\sigma(t, q^i, \dot{q}^i) = 0 \quad \sigma = 1, \dots, n - r$$

with $g^\sigma \in \mathfrak{F}(j_1(\mathcal{V}_{n+1}))$ and $\text{rank} \left\| \frac{\partial(g^1 \dots g^{n-r})}{\partial(\dot{q}^1 \dots \dot{q}^n)} \right\| = n - r$.

The geometric setup developed in § 3.1 is recovered regarding the intrinsic lagrangian as the pull-back of the extrinsic one. In coordinates, the resulting state of affairs is summarized into the equations

$$L(t, q^i, z^A) = \mathcal{L}(t, q^i, \psi^i(t, q^i, z^A)) \quad (3.13a)$$

$$g^\sigma(t, q^i, \psi^i(t, q^i, z^A)) = 0 \quad (3.13b)$$

These, in turn, imply the relations

$$\frac{\partial L}{\partial q^i} = \frac{\partial \mathcal{L}}{\partial q^i} + \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial q^i}, \quad \frac{\partial L}{\partial z^A} = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial z^A} \quad (3.14a)$$

$$\frac{\partial g^\sigma}{\partial q^i} + \frac{\partial g^\sigma}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial q^i} = 0, \quad \frac{\partial g^\sigma}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial z^A} = 0 \quad (3.14b)$$

Let us now consider the product manifold $\mathcal{E} := \mathcal{V}_{n+1} \times \mathbb{R}^{n-r}$, referred to fiber coordinates t, q^i, λ_σ . The natural projection $\mathcal{E} \xrightarrow{\pi_1} \mathcal{V}_{n+1}$ makes \mathcal{E} into a vector bundle over \mathcal{V}_{n+1} and therefore also into a fiber bundle over \mathbb{R} . We regard \mathcal{E} as the event space of a fictitious *unconstrained* system, glue the functions $\mathcal{L}, g^\sigma \in$

$\mathfrak{F}(j_1(\mathcal{V}_{n+1}))$ into a single function $\hat{\mathcal{L}} := \mathcal{L} + \lambda_\sigma g^\sigma \in \mathfrak{F}(j_1(\mathcal{E}))$ and adopt $\hat{\mathcal{L}}$ as a (singular) lagrangian for the newer system.

In this way, proceeding as in § 3.1, to every continuous, piecewise differentiable section $\mu : [t_0, t_1] \rightarrow \mathcal{E}$ we associate the action integral

$$I[\mu] := \int_{\hat{\mu}} \hat{\mathcal{L}} dt := \sum_{s=1}^N \int_{a_{s-1}}^{a_s} \left[\mathcal{L}(t, q^i, \dot{q}^i) + \lambda_\sigma g^\sigma(t, q^i, \dot{q}^i) \right] dt \quad (3.15)$$

$\hat{\mu}$ denoting the lift of μ to a section $\hat{\mu} : [t_0, t_1] \rightarrow j_1(\mathcal{E})$. We can then state

Theorem 3.2 *The projection $\pi_1 : \mathcal{E} \rightarrow \mathcal{V}_{n+1}$ sets up a 1–1 correspondence between extremals of the functional (3.15) and extremals of the functional (3.1) satisfying condition a) of Theorem 3.1.*

Proof. According to elementary (holonomic) variational calculus, the extremals of the functional (3.15) are determined by the Euler–Lagrange equations

$$g^\sigma(t, q^i, \dot{q}^i) = 0 \quad (3.16a)$$

$$\frac{d}{dt} \frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^i} - \frac{\partial \hat{\mathcal{L}}}{\partial q^i} = 0 \quad (3.16b)$$

completed by the Erdmann–Weierstrass corner conditions, asserting the continuity of the “momenta” $\frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^i}$ and of the “hamiltonian” $(\dot{q}^i \frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^i} - \hat{\mathcal{L}})$ along $\hat{\mu}$.

On the other hand, eqs. (3.14a,b) and the definition of $\hat{\mathcal{L}}$ imply the relations

$$\frac{\partial L}{\partial q^i} = \frac{\partial \hat{\mathcal{L}}}{\partial q^i} + \frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^j} \frac{\partial \psi^j}{\partial q^i}, \quad \frac{\partial L}{\partial z^A} = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^j} \frac{\partial \psi^j}{\partial z^A} \quad (3.17)$$

From these, taking eqs. (3.13a,b), (3.14a,b) into account, we conclude:

- if $\mu : q^i = q^i(t), \lambda_\sigma = \lambda_\sigma(t)$ is an extremal for the functional (3.15), the functions $q^i(t)$ and $p_i(t) := \hat{\mu}^* \left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^i} \right)$ satisfy eqs. (3.10a,b,c), (3.11). Therefore, the class $\wp(\pi_1 \cdot \mu)$ associated with the section $\pi_1 \cdot \mu : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ is non-empty, since it contains at least the 1-form $\hat{\lambda} = \hat{\mu}^* \left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}^i} \right) \hat{\omega}^i$;
- conversely, let $q^i = q^i(t), p_i = p_i(t)$ be any solution of eqs. (3.10a,b,c), (3.11); then eqs. (3.10a), (3.13b) automatically imply eq. (3.16a). Moreover, on account of the identification (3.13a), eq. (3.10c) reads

$$0 = p_i \frac{\partial \psi^i}{\partial z^A} - \frac{\partial L}{\partial z^A} = \left(p_i - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \frac{\partial \psi^i}{\partial z^A}$$

In view of the assumption $\text{rank} \left\| \frac{\partial(g^1 \dots g^{n-r})}{\partial(\dot{q}^1 \dots \dot{q}^n)} \right\| = n - r$, the second equation (3.14b) ensures that the linear system

$$\frac{\partial g^\sigma}{\partial \dot{q}^i} \lambda_\sigma = p_i - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \quad (3.18a)$$

can be uniquely solved for the functions $\lambda_\sigma = \lambda_\sigma(t)$.

Together with eqs. (3.14a,b), this implies the further relation

$$p_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^k} \right) \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} = -\frac{\partial \mathcal{L}}{\partial q^i} - \lambda_\sigma \frac{\partial g^\sigma}{\partial q^i} \quad (3.18b)$$

Eqs. (3.18a,b) allow to cast eq. (3.10b) in the equivalent form

$$0 = \frac{dp_i}{dt} + p_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} + \lambda_\sigma \frac{\partial g^\sigma}{\partial q^i}$$

reproducing the content of eqs. (3.16b), with $\hat{\mathcal{L}} = \mathcal{L} + \lambda_\sigma g^\sigma$.

Finally, on account of eq. (3.18a), eqs. (3.11) are easily seen to coincide with the Erdmann–Weierstrass conditions along the curve $q^i = q^i(t)$, $\lambda_\sigma = \lambda_\sigma(t)$. Every solution $q^i = q^i(t)$, $p_i = p_i(t)$ of the system (3.10a,b,c), (3.11) is therefore related to a unique extremal of the functional (3.15) by the projection algorithm. \square

3.3 Pontryagin’s “maximum principle” revisited

From a mathematical viewpoint, eqs. (3.10a,b,c), form a set of $2n + r$ equations for the $2n + r$ unknowns $q^i(t)$, $z^A(t)$, $p_i(t)$. In this respect, they have a natural setting in the geometrical environment provided by the contact bundle $\mathcal{C}(\mathcal{A})$.

As pointed out in Appendix A, the latter is a vector bundle over \mathcal{A} , identical to the pull-back of the phase space $V^*(\mathcal{V}_{n+1})$ through the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\kappa}} & V^*(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array}$$

Accordingly, we refer $\mathcal{C}(\mathcal{A})$ to fibered coordinates t, q^i, z^A, p_i , with t, q^i, z^A coordinates in \mathcal{A} and t, q^i, p_i coordinates in $V^*(\mathcal{V}_{n+1})$.

The advantage of the environment $\mathcal{C}(\mathcal{A})$ comes from the presence of a distinguished *Liouville 1-form* $\tilde{\Theta}$, expressed in coordinates as

$$\tilde{\Theta} = p_i \tilde{\omega}^i = p_i (dq^i - \psi^i dt) \quad (3.19)$$

(For simplicity, we are not distinguishing between covariant objects in \mathcal{A} and their pull-back in $\mathcal{C}(\mathcal{A})$, namely we are writing ψ^i for $\zeta^*(\psi^i)$, $\tilde{\omega}^i$ for $\zeta^*(\tilde{\omega}^i)$ etc.)

By means of $\tilde{\Theta}$, every Lagrangian $L \in \mathfrak{F}(\mathcal{A})$ may be lifted to a 1-form ϑ_L over $\mathcal{C}(\mathcal{A})$ according to the prescription

$$\vartheta_L := L dt + \tilde{\Theta} = (L - p_i \psi^i) dt + p_i dq^i := -\mathcal{H} dt + p_i dq^i \quad (3.20)$$

The difference $\mathcal{H} := p_i \psi^i - L$ is known in the literature as the *Pontryagin Hamiltonian*. Needless to say, \mathcal{H} is not a Hamiltonian in the traditional sense, but a function on the contact bundle.

To understand the role of the ϑ_L we focus on the fibration $\mathcal{C}(\mathcal{A}) \xrightarrow{v} \mathcal{V}_{n+1}$ given by the composite map $v := \pi \cdot \hat{\kappa}$. A piecewise differentiable section $(\sigma, [t_0, t_1])$ consisting of a finite family of closed arcs

$$\sigma^{(s)} : [a_{s-1}, a_s] \rightarrow \mathcal{C}(\mathcal{A}), \quad s = 1, \dots, N, \quad t_0 = a_0 < a_1 < \dots < a_N = t_1$$

will be called *v-continuous* if and only if the composite map $v \cdot \sigma$ is continuous, i.e. if and only if σ projects onto a continuous, piecewise differentiable section $v \cdot \sigma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$. A deformation $\sigma_\xi = \{(\sigma_\xi^{(s)}, [a_{s-1}(\xi), a_s(\xi)])\}$ will similarly be called *v-continuous* if and only if all sections σ_ξ are *v-continuous*. A necessary and sufficient condition for this to happen is the validity of the *matching conditions* (2.40), synthetically written as

$$\lim_{t \rightarrow a_s^+(\xi)} v \cdot \sigma_\xi(t) = \lim_{t \rightarrow a_s^-(\xi)} v \cdot \sigma_\xi(t) \quad s = 1, \dots, N-1 \quad (3.21)$$

A *v-continuous* deformation σ_ξ is said to preserve the end points of $v \cdot \sigma$ if and only if $v \cdot \sigma_\xi$ is a deformation with fixed endpoints. A vector field along σ tangent to the orbits of a *v-continuous* deformation is called an *infinitesimal deformation*.

Notice that, since the stated definitions do not include any admissibility requirement for the sections $v \cdot \sigma_\xi$, the only condition needed in order for a vector field $X^i \left(\frac{\partial}{\partial q^i} \right)_\sigma + \Gamma^A \left(\frac{\partial}{\partial z^A} \right)_\sigma + \Pi_i \left(\frac{\partial}{\partial p_i} \right)_\sigma$ to represent an infinitesimal deformation of σ is the consistency with the matching conditions (3.21), expressed in components by the *jump relations*

$$\lim_{t \rightarrow a_s^+(\xi)} \left(X^i + \alpha_s \frac{dq^i}{dt} \right) = \lim_{t \rightarrow a_s^-(\xi)} \left(X^i + \alpha_s \frac{dq^i}{dt} \right) \quad s = 1, \dots, N-1 \quad (3.22)$$

with $\alpha_s = \left(\frac{da_s}{d\xi} \right)_{\xi=0}$.

By means of ϑ_L we now define an action integral over $\mathcal{C}(\mathcal{A})$, assigning to each *v-continuous* section $\sigma : q^i = q^i(t), z^A = z^A(t), p_i = p_i(t)$ the real number

$$\mathcal{I}[\sigma] := \int_\sigma \vartheta_L = \int_{t_0}^{t_1} \left(p_i \frac{dq^i}{dt} - \mathcal{H} \right) dt \quad (3.23)$$

For any *v-continuous* deformations σ_ξ preserving the end points of $v \cdot \sigma$ we have then the relation

$$\begin{aligned} \frac{d\mathcal{I}[\sigma_\xi]}{d\xi} \Big|_{\xi=0} &= \int_{t_0}^{t_1} \left[\left(\frac{dq^i}{dt} - \frac{\partial \mathcal{H}}{\partial p_i} \right) \Pi_i - \left(\frac{dp_i}{dt} + \frac{\partial \mathcal{H}}{\partial q^i} \right) X^i - \frac{\partial \mathcal{H}}{\partial z^A} \Gamma^A \right] dt + \\ &+ \sum_{s=1}^N \left\{ \lim_{t \rightarrow a_s^-} \left[\alpha_s \left(p_i \frac{dq^i}{dt} - \mathcal{H} \right) + p_i X^i \right] - \lim_{t \rightarrow a_{s-1}^+} \left[\alpha_{s-1} \left(p_i \frac{dq^i}{dt} - \mathcal{H} \right) + p_i X^i \right] \right\} \end{aligned}$$

From the latter, taking eqs. (3.22) and the conditions $X^i(t_0) = X^i(t_1) = 0$ into account, we conclude that the vanishing of $\frac{d\mathcal{I}}{d\xi} \Big|_{\xi=0}$ under arbitrary deformations of the given class is mathematically equivalent to the system

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i}; \quad \frac{\partial \mathcal{H}}{\partial z^A} = 0 \quad (3.24a)$$

completed with the continuity conditions

$$[p_i]_{a_s} = [\mathcal{H}]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (3.24b)$$

where, as usual, we are denoting by $[f]_{a_s}$ the jump of the function $f(t)$ at $t = a_s$.

In view of the definition of the Pontryagin Hamiltonian \mathcal{H} , eqs. (3.24a,b) are easily seen to reproduce the content of eqs. (3.10a,b,c), (3.11), the continuity requirement $[q^i]_{a_s} = 0$ being implicit in the definition of σ .

As far as the *ordinary* extremals are concerned, the original (constrained) variational problem in the event space is therefore equivalent to a *free* variational problem in the contact manifold. This is precisely the essence of Pontryagin's *maximum principle*.

As a further comment on eqs. (3.24), let us digress on the special situation determined by the ansatz $L = 0$. Under the stated circumstance, the functional

$$\mathcal{I}_0[\sigma] := \int_{\sigma} \tilde{\Theta} = \int_{t_0}^{t_1} p_i \left(\frac{dq^i}{dt} - \psi^i \right) dt \quad (3.25)$$

is an intrinsic attribute of the manifold $\mathcal{C}(\mathcal{A})$, entirely determined by the Liouville 1-form (3.19). Its role is clarified by the following

Proposition 3.1 *Let $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ denote any continuous, piecewise differentiable section. Then:*

- a) *γ is admissible if and only if the functional (3.25) admits at least one extremal σ projecting onto γ , i.e. satisfying $v \cdot \sigma = \gamma$;*
- b) *for any such γ , the totality of extremals of \mathcal{I}_0 projecting onto γ form a finite dimensional vector space over \mathbb{R} , with dimension equal to the abnormality index of γ .*

Proof. For $L = 0$, eqs. (3.24a,b) reduce to

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A) \quad (3.26a)$$

$$\frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i} p_k = 0 \quad (3.26b)$$

$$p_i \frac{\partial \psi^i}{\partial z^A} = 0 \quad (3.26c)$$

$$[p_i]_{a_s} = [p_i \psi^i]_{a_s} = 0 \quad s = 1, \dots, N-1 \quad (3.26d)$$

Eq. (3.26a) is the admissibility requirement for the section $v \cdot \sigma$. Due to this fact, if an extremal σ of the functional (3.25) satisfies $v \cdot \sigma = \gamma$, its projection $\zeta \cdot \sigma$ under the map $\zeta : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$ coincides with the lift $\hat{\gamma} : [t_0, t_1] \rightarrow \mathcal{A}$.

For any admissible γ , the extremals projecting onto γ are therefore in 1–1 correspondence with the solutions $p_i(t)$ of the homogeneous system (3.26b,c,d),

with the functions $q^i(t), z^A(t)$ regarded as given. On the other hand, according to Proposition 2.5, eqs. (3.26b,c,d) are precisely the relations characterizing the totality of virtual 1-forms $p_i(t)\dot{\omega}^i$ belonging to the annihilator $(\Upsilon(\mathfrak{W}))^0$.

Both assertions a) and b) follow easily from this fact. \square

In the language of § 2.5, Proposition 3.1 asserts that a section $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ describes a *normal* evolution of the system if and only if the functional (3.25) admits *exactly one* extremal projecting onto γ , namely the one corresponding to the trivial solution $p_i(t) = 0$. If the extremals projecting onto γ are more than one, γ represents an *abnormal* evolution; if no such extremal exists, γ is not admissible.

Returning to the action integral (3.23) we can now state

Proposition 3.2 *The totality of extremals of the functional (3.23) projecting onto a section $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ is an affine space, modelled on the vector space formed by the extremals of the functional (3.25) projecting onto γ .*

The proof, entirely straightforward, is left to the reader.

The previous arguments provide a restatement of Theorem 3.1 in the environment $\mathcal{C}(\mathcal{A})$. In particular, it is worth remarking that, in general, the projection algorithm $\sigma \rightarrow v \cdot \sigma$, applied to the totality of extremals of the functional (3.23), does not yield back *all* the extremals of the functional (3.1), but only a subclass, wide enough to include the *ordinary* ones. The missing extremals may be obtained determining the abnormal evolutions by means of Proposition 3.1, finding out which ones have an exceptional character, and analysing each of them individually.

3.4 Hamiltonian formulation

As pointed out in § 3.2, all ordinary extremals of the functional (3.1) are projections of extremals of the functional (3.23). Let us analyse the implications of this fact.

To this end, temporarily leaving aside all aspects related to the presence of corners, we observe that a differentiable curve σ in $\mathcal{C}(\mathcal{A})$ is at the same time a section with respect to the fibration $\mathcal{C}(\mathcal{A}) \xrightarrow{t} \mathbb{R}$ and an extremal for the functional (3.23) if and only if its tangent vector field $Z := \sigma_*\left(\frac{\partial}{\partial t}\right)$ satisfies the properties

$$\langle Z, dt \rangle = 1, \quad Z \lrcorner d\vartheta_L = 0 \quad (3.27)$$

On account of eq. (3.20), at any $\varsigma \in \mathcal{C}(\mathcal{A})$ a necessary and sufficient condition for the existence of at least one vector $Z \in T_\varsigma(\mathcal{C}(\mathcal{A}))$ satisfying eqs. (3.27) is the validity of the relations

$$\left(\frac{\partial \mathcal{H}}{\partial z^A}\right)_\varsigma = 0 \quad (3.28a)$$

Points ς at which eqs. (3.27) admit a *unique* solution Z will be called *regular points* for the functional (3.23). In coordinates, the regularity requirement is expressed by the condition

$$\det \left(\frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B} \right)_\varsigma \neq 0 \quad (3.28b)$$

In view of eq. (3.28b), in a neighborhood of each regular point eqs. (3.28a) may be solved for the z^A 's, giving rise to a representation of the form

$$z^A = z^A(t, q^1, \dots, q^n, p_1, \dots, p_n) \quad (3.29)$$

The regular points form therefore a $(2n+1)$ -dimensional submanifold $\mathcal{S} \xrightarrow{j} \mathcal{C}(\mathcal{A})$, locally diffeomorphic to the phase space $V^*(\mathcal{V}_{n+1})$.

Inserting eqs. (3.29) into the first pair of relations (3.24a) gives rise to a system of ordinary differential equations in normal form for the unknowns $q^i(t), p_i(t)$. The algorithm is readily implemented denoting by $H := j^*(\mathcal{H})$ the pull-back of the Pontryagin Hamiltonian to the submanifold \mathcal{S} , expressed in coordinates as

$$H = \mathcal{H}(t, q^r, z^A(t, q^i, p_i), p_r) = p_k \psi^k(t, q^r, z^A(t, q^i, p_i)) - L(t, q^r, z^A(t, q^i, p_i))$$

In view of eqs. (3.20), (3.28a) we have then the identifications

$$\frac{\partial H}{\partial p_i} = \frac{\partial \mathcal{H}}{\partial p_i} + \frac{\partial \mathcal{H}}{\partial z^A} \frac{\partial z^A}{\partial p_i} = \psi^i \quad (3.30a)$$

$$\frac{\partial H}{\partial q^i} = \frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial z^A} \frac{\partial z^A}{\partial q^i} = p_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} \quad (3.30b)$$

On account of these, the first pair of equations (3.24a) takes the final form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad (3.31a)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad (3.31b)$$

The original *constrained* Lagrangian variational problem has thus been reduced to a *free* Hamiltonian problem on the submanifold $j : \mathcal{S} \rightarrow \mathcal{C}(\mathcal{A})$, with Hamiltonian $H(t, q^1, \dots, q^n, p_1, \dots, p_n)$ identical to the pull-back $H = j^*(\mathcal{H})$. Conversely, setting $H = j^*(\mathcal{H})$, the inverse Legendre transformation $\dot{q}^i = \frac{\partial H}{\partial p_i}$, together with eq. (3.30a), yields back the constraint equations $\dot{q}^i = \psi^i(t, q^k, z^A)$. Once again, all this is in full agreement with Pontryagin's principle.

A v -continuous extremal of the functional (3.23) consisting of a finite family of closed arcs $\sigma^{(s)} : [a_{s-1}, a_s] \rightarrow \mathcal{C}(\mathcal{A})$, each contained in (a connected component of) the submanifold \mathcal{S} will be called a *regular extremal*.

Singular extremals, partly, or even totally lying outside \mathcal{S} may also exist. In fact, while eq. (3.28a) is part of the system (3.24a,b), and must therefore be satisfied by any extremal, the requirement (3.28b) has only to do with the *well-posedness* of the Cauchy problem for the subsystem (3.24a).

On the other hand, by construction, the Hamilton equations (3.31a,b) determines only the regular extremals. The singular ones, when present, have therefore to be dealt with directly, looking for solutions of eqs. (3.24a,b) not arising from a well posed Cauchy problem. In principle, this could be done extending to the

non-holonomic context the concepts and methods commonly adopted in the study of singular Lagrangians [27]. The argument is beyond the purposes of the present work, and will not be pursued.

To complete our analysis, let us finally discuss the role of eqs. (3.24b) in the study of *corners*. To this end, we consider the cotangent space $T^*(\mathcal{V}_{n+1})$, referred to local fibered coordinates t, q^i, p_0, p_i , and denote by

$$\vartheta := p_0 dt + p_i dq^i \quad (3.32)$$

the corresponding Liouville 1-form. We also recall (see Appendix A) that the equivalence relation (A.6) determines a fibration $T^*(\mathcal{V}_{n+1}) \rightarrow V^*(\mathcal{V}_{n+1})$, expressed in coordinates as $(t, q^i, p_0, p_i) \mapsto (t, q^i, p_i)$.

Making use of the 1-forms (3.20), (3.32) let us now introduce a morphism $\mathcal{C}(\mathcal{A}) \xrightarrow{\Psi} T^*(\mathcal{V}_{n+1})$ fibered over $V^*(\mathcal{V}_{n+1})$, based on the prescription

$$\Psi^*(\vartheta) = \vartheta_L$$

In coordinates, we have the explicit representation

$$\Psi : \quad t = t, \quad q^i = q^i, \quad p_i = p_i, \quad p_0 = -\mathcal{H}(t, q^i, p_i, z^A) \quad (3.33)$$

The content of eqs. (3.24b) is then summarized into the following

Proposition 3.3 *For any v -continuous extremal $(\sigma, [t_0, t_1])$ of the functional (3.23), the composite map $\Psi \cdot \sigma : [t_0, t_1] \rightarrow T^*(\mathcal{V}_{n+1})$ is necessarily continuous.*

The previous arguments provide a simple characterization of the jumps that may possibly occur along a *regular* extremal $\sigma : \{(\sigma^{(s)}, [a_{s-1}, a_s])\}$. To this end we observe that the restriction of the map (3.33) to the submanifold $\mathcal{S} \subset \mathcal{C}(\mathcal{A})$ determines an immersion $\Psi : \mathcal{S} \rightarrow T^*(\mathcal{V}_{n+1})$ and that, as already pointed out, at each $\varsigma \in \mathcal{S}$ there exists, locally, one and only one differentiable extremal of the functional (3.23) through ς .

On the other hand, by Proposition 3.3, for each $s = 1, \dots, N-1$, the arcs $\sigma^{(s)}$ and $\sigma^{(s+1)}$ are related by the condition $\Psi(\sigma^{(s)}(a_s)) = \Psi(\sigma^{(s+1)}(a_s))$. From this it is readily seen that the admissible discontinuities of σ or, what is the same, the admissible corners in the projection $\gamma := v \cdot \sigma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ may only occur at points in which the immersion $\Psi : \mathcal{S} \rightarrow T^*(\mathcal{V}_{n+1})$ is not injective.

A Geometry of the velocity space

(i) Given the event space \mathcal{V}_{n+1} , let $j_1(\mathcal{V}_{n+1})$ and $V(\mathcal{V}_{n+1})$ respectively denote the first-jet bundle and the vertical bundle associated with the fibration $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$. Both spaces $j_1(\mathcal{V}_{n+1})$ and $V(\mathcal{V}_{n+1})$ may be viewed as submanifolds of the tangent bundle $T(\mathcal{V}_{n+1})$ according to the identifications

$$j_1(\mathcal{V}_{n+1}) = \{z \mid z \in T(\mathcal{V}_{n+1}), \langle z, dt \rangle = 1\} \quad (A.1a)$$

$$V(\mathcal{V}_{n+1}) = \{v \mid v \in T(\mathcal{V}_{n+1}), \langle v, dt \rangle = 0\} \quad (A.1b)$$

Eqs. (A.1a,b) point out the nature of $j_1(\mathcal{V}_{n+1})$ as an affine bundle over \mathcal{V}_{n+1} , modelled on the vertical space $V(\mathcal{V}_{n+1})$ [22, 26, 17, 16].

Given any local fibered coordinate system t, q^i over \mathcal{V}_{n+1} , we respectively refer $j_1(\mathcal{V}_{n+1})$ to jet-coordinates t, q^i, \dot{q}^i and $V(\mathcal{V}_{n+1})$ to coordinates t, q^i, u^i . In terms of these, the content of eqs. (A.1a,b) is summarized into the relations

$$z = \left(\frac{\partial}{\partial t} + \dot{q}^i(z) \frac{\partial}{\partial q^i} \right)_{\pi(z)} \quad \forall z \in j_1(\mathcal{V}_{n+1}) \quad (\text{A.2a})$$

$$v = u^i(v) \left(\frac{\partial}{\partial q^i} \right)_{\pi(v)} \quad \forall v \in V(\mathcal{V}_{n+1}) \quad (\text{A.2b})$$

Eq. (A.2a) allows to set up a vector-bundle homomorphism

$$\begin{array}{ccc} T(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\nu} & V(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array}$$

assigning to each $X \in T_z(j_1(\mathcal{V}_{n+1}))$ a vertical vector $\nu(X) \in V_{\pi(z)}(\mathcal{V}_{n+1})$ according to the prescription

$$\nu(X) := \pi_*(X) - \left\langle \pi_*(X), (dt)_{\pi(z)} \right\rangle z \quad (\text{A.3})$$

In coordinates, introducing the notation

$$\omega^i := dq^i - \dot{q}^i dt \quad (\text{A.4})$$

eq. (A.3) takes the explicit form

$$\nu(X) = \left\langle X, \omega^i|_z \right\rangle \left(\frac{\partial}{\partial q^i} \right)_{\pi(z)} \quad (\text{A.5})$$

(ii) The dual of the vertical bundle, henceforth denoted by $V^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$, is called the *phase space*. In view of eq. (A.1b), the latter is canonically isomorphic to the quotient of the cotangent space $T^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ by the equivalence relation

$$\sigma \sim \sigma' \iff \begin{cases} \pi(\sigma) = \pi(\sigma') \\ \sigma - \sigma' \propto dt|_{\pi(\sigma)} \end{cases} \quad (\text{A.6})$$

For simplicity, we preserve the notation \langle , \rangle for the pairing between $V(\mathcal{V}_{n+1})$ and $V^*(\mathcal{V}_{n+1})$.

Every local coordinate system t, q^i in \mathcal{V}_{n+1} induces fibered coordinates t, q^i, p_i in $V^*(\mathcal{V}_{n+1})$, with $p_i(\hat{\sigma}) := \left\langle \hat{\sigma}, \left(\frac{\partial}{\partial q^i} \right)_{\pi(\hat{\sigma})} \right\rangle \forall \hat{\sigma} \in V^*(\mathcal{V}_{n+1})$.

(iii) Let $V(j_1(\mathcal{V}_{n+1})) \xrightarrow{\zeta} j_1(\mathcal{V}_{n+1})$ denote the vertical bundle over $j_1(\mathcal{V}_{n+1})$ relative to the fibration $\pi : j_1(\mathcal{V}_{n+1}) \rightarrow \mathcal{V}_{n+1}$. Given any jet coordinate system t, q^i, \dot{q}^i in

$j_1(\mathcal{V}_{n+1})$, we refer $V(j_1(\mathcal{V}_{n+1}))$ to fibered coordinates $t, q^i, \dot{q}^i, \dot{u}^i$, with \dot{u}^i uniquely defined by the requirement

$$W = \dot{u}^i(W) \left(\frac{\partial}{\partial \dot{q}^i} \right)_{\zeta(W)} \quad \forall W \in V(j_1(\mathcal{V}_{n+1}))$$

The affine character of the fibration $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ provides a canonical identification of $V(j_1(\mathcal{V}_{n+1}))$ with the pull-back of the vertical bundle $V(\mathcal{V}_{n+1})$. This gives rise to a vector bundle homomorphism

$$\begin{array}{ccc} V(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\varrho} & V(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \quad (\text{A.7a})$$

expressed in coordinates as

$$\varrho \left(W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_z \right) = W^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_{\pi(z)} \quad (\text{A.7b})$$

The result is well known from non-holonomic mechanics (see e.g [16, 17, 22, 26] and references therein). A sketchy proof may be traced as follows: for each $z \in j_1(\mathcal{V}_{n+1})$, the fiber $\Sigma_z := \pi^{-1}(\pi(z))$ through z is an affine submanifold of $j_1(\mathcal{V}_{n+1})$, modelled on the vertical space $V_{\pi(z)}(\mathcal{V}_{n+1})$. Every pair (z, v) , $v \in V_{\pi(z)}(\mathcal{V}_{n+1})$ is therefore an “applied vector” at z in Σ_z , i.e. an element of the tangent space $T_z(\Sigma_z)$. On the other hand, by definition, $T_z(\Sigma_z)$ is isomorphic to the vertical space $V_z(j_1(\mathcal{V}_{n+1}))$. By varying z we conclude that the totality of pairs $(z, v) \in j_1(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} V(\mathcal{V}_{n+1})$ is in 1–1 correspondence with the points of $V(j_1(\mathcal{V}_{n+1}))$, thus establishing the result.

(iv) The pull-back of the space $V^*(\mathcal{V}_{n+1})$ through the map $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ determines another important space $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, henceforth referred to as the *contact bundle*. Once again, the situation is conveniently summarized into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\kappa} & V^*(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \quad (\text{A.8})$$

Notice that, by construction, $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ is at the same time a vector bundle over $j_1(\mathcal{V}_{n+1})$ and an affine bundle over $V^*(\mathcal{V}_{n+1})$.

The manifold $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ will be referred to coordinates t, q^i, \dot{q}^i, p_i related in an obvious way to the coordinates t, q^i, \dot{q}^i in $j_1(\mathcal{V}_{n+1})$ and t, q^i, p_i in $V^*(\mathcal{V}_{n+1})$. Every $\sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1}))$ will be called a *contact 1-form* over $j_1(\mathcal{V}_{n+1})$.

In view of the stated definition, a contact 1-form σ is essentially a pair $(z, \hat{\sigma}) \in j_1(\mathcal{V}_{n+1}) \times V^*(\mathcal{V}_{n+1})$, with $\hat{\sigma} \in V_{\pi(z)}^*(\mathcal{V}_{n+1})$. Now, by eq. (A.3), every such pair

determines a linear functional on the tangent space $T_z(j_1(\mathcal{V}_{n+1}))$ according to the prescription

$$\langle \sigma, X \rangle := \langle \hat{\sigma}, \nu(X) \rangle \quad \forall X \in T_z(j_1(\mathcal{V}_{n+1})) \quad (\text{A.9})$$

In coordinates, recalling eq. (A.5), the definition of $p_i(\hat{\sigma})$ and the identification $p_i(\sigma) = p_i(\hat{\sigma})$, eq. (A.9) takes the explicit form

$$\langle \sigma, X \rangle = \left\langle \hat{\sigma}, \left(\frac{\partial}{\partial q^i} \right)_{\pi(z)} \right\rangle \langle \omega^i|_z, X \rangle = \langle p_i(\hat{\sigma}) \omega^i|_z, X \rangle = \langle p_i(\sigma) \omega^i|_{\zeta(\sigma)}, X \rangle$$

From the latter it is easily seen that the knowledge of the functional (A.9) is mathematically equivalent to the knowledge of σ .

The contact bundle $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ is therefore identical to the vector subbundle of the cotangent space $T^*(j_1(\mathcal{V}_{n+1}))$ locally generated by the 1-forms (A.4), while the coordinates p_i coincide with the components involved in the representation

$$\sigma = p_i(\sigma) \omega^i|_{\zeta(\sigma)} \quad \forall \sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1})) \quad (\text{A.10})$$

An important attribute of the contact bundle is the presence of a distinguished *Liouville 1-form* Θ , assigning to each $\sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1}))$ the linear functional $\Theta|_\sigma \in T_\sigma^*[\mathcal{C}(j_1(\mathcal{V}_{n+1}))]$ defined by the prescription

$$\langle \Theta|_\sigma, Y \rangle := \langle \sigma, \zeta_*(Y) \rangle \quad \forall Y \in T_\sigma[\mathcal{C}(j_1(\mathcal{V}_{n+1}))] \quad (\text{A.11})$$

In coordinates, using the same notation ω^i for the 1-forms (A.4) and for their pull-back on $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, eqs. (A.10), (A.11) yield the representation

$$\Theta = p_i \omega^i = p_i(dq^i - \dot{q}^i dt) \quad (\text{A.12})$$

(v) In the presence of non-holonomic constraints, let $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ denote the submanifold of $j_1(\mathcal{V}_{n+1})$ formed by the totality of *admissible velocities*.

As in § 2.1, we refer \mathcal{A} to fibered coordinates t, q^i, z^A , and represent the imbedding $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ locally as

$$\dot{q}^i = \psi^i(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad i = 1, \dots, n$$

The concepts of vertical vector and contact 1-form are immediately extended to the submanifold \mathcal{A} : as usual, the vertical bundle $V(\mathcal{A})$ is the kernel of the push-forward $T(\mathcal{A}) \xrightarrow{\pi_*} T(\mathcal{V}_{n+1})$, while the contact bundle $\mathcal{C}(\mathcal{A})$ is the pull-back on \mathcal{A} of the contact bundle $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, as expressed by the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{i} & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \\ \zeta \downarrow & & \downarrow \zeta \\ \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \end{array} \quad (\text{A.13})$$

The manifolds $V(\mathcal{A})$, $\mathcal{C}(\mathcal{A})$ will be respectively referred to coordinates t, q^i, z^A, v^A and t, q^i, z^A, p_i . In this way, setting

$$\tilde{\omega}^i := i^*(\omega^i) = dq^i - \psi^i(t, q^k, z^A) dt \quad (\text{A.14})$$

we have the representations $X = v^A(X) \left(\frac{\partial}{\partial z^A} \right)_{\zeta(X)}$, $\sigma = p_i(\sigma) \tilde{\omega}^i|_{\zeta(\sigma)} \forall X \in V(\mathcal{A})$, $\sigma \in \mathcal{C}(\mathcal{A})$.

The pull-back $\tilde{\Theta} := \hat{i}^*(\Theta)$ of the Liouville 1-form of $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ will be called the Liouville 1-form of \mathcal{A} . In coordinates, eqs. (A.12), (A.14) provide the representation

$$\tilde{\Theta} = p_i \tilde{\omega}^i = p_i (dq^i - \psi^i dt) \quad (\text{A.15})$$

The restriction of the push-forward $i_* : T(\mathcal{A}) \rightarrow T(j_1(\mathcal{V}_{n+1}))$ to the vertical subspace $V(\mathcal{A})$ determines a vector bundle homomorphism

$$\begin{array}{ccc} V(\mathcal{A}) & \xrightarrow{i_*} & V(j_1(\mathcal{V}_{n+1})) \\ \zeta \downarrow & & \downarrow \zeta \\ \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \end{array}$$

Composing with diagram (A.7a) and setting $\hat{\varrho} := \varrho \cdot i_*$, we get a further homomorphism

$$\begin{array}{ccc} V(\mathcal{A}) & \xrightarrow{\hat{\varrho}} & V(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \quad (\text{A.16})$$

In coordinates, eqs. (A.7b), (A.16) provide the representation

$$\hat{\varrho} \left[V^A \left(\frac{\partial}{\partial z^A} \right)_z \right] = \varrho \left[V^A \left(\frac{\partial \psi^i}{\partial z^A} \right)_z \left(\frac{\partial}{\partial q^i} \right)_{i(z)} \right] = V^A \left(\frac{\partial \psi^i}{\partial z^A} \right)_z \left(\frac{\partial}{\partial q^i} \right)_{\pi(z)} \quad (\text{A.17})$$

Finally, by diagrams (A.13), (A.8), setting $\hat{\kappa} := \kappa \cdot \hat{i}$ we get a bundle morphism

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\kappa}} & V^*(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \quad (\text{A.18})$$

described in coordinates as

$$t(\hat{\kappa}(\sigma)) = t(\sigma), \quad q^i(\hat{\kappa}(\sigma)) = q^i(\sigma), \quad p_i(\hat{\kappa}(\sigma)) = p_i(\sigma)$$

The latter allows to regard the contact bundle $\mathcal{C}(\mathcal{A})$ as a fibre bundle over the phase space $V^*(\mathcal{V}_{n+1})$, identical to the pull-back of $V^*(\mathcal{V}_{n+1})$ through the map $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$.

B Finite deformations with fixed end points: an existence theorem

(i) Given an admissible, piecewise differentiable section $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$, a crucial question is establishing under what circumstances *every* admissible infinitesimal deformation vanishing at the end points of γ is tangent to an admissible finite deformation γ_ξ with fixed end points. The following preliminaries help simplifying the discussion.

Lemma B.1 *Let $\hat{\gamma} : (c, d) \rightarrow \mathcal{A}$ be the lift of an admissible differentiable section $\gamma : (c, d) \rightarrow \mathcal{V}_{n+1}$. Then, for any closed interval $[a, b] \subset (c, d)$ there exists a fibered local chart (U, k) , $k = (t, q^1, \dots, q^n, z^1, \dots, z^r)$ satisfying the properties*

$$\hat{\gamma}(t) \in U \quad \forall t \in [a, b]; \quad (\text{B.1a})$$

$$\text{the intersection } \hat{\gamma}((c, d)) \cap U \text{ coincides with the curve } q^i = z^A = 0; \quad (\text{B.1b})$$

$$\psi^i(\hat{\gamma}(t)) = \left(\frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}(t)} = 0 \quad \forall \hat{\gamma}(t) \in U. \quad (\text{B.1c})$$

Proof. A straightforward argument, regarded as known, ensures the existence of local charts (V, h) , $h = (t, \bar{q}^1, \dots, \bar{q}^n)$ in \mathcal{V}_{n+1} and (W, k') , $k' = (t, x^1, \dots, x^{n+r})$ in \mathcal{A} satisfying the conditions

$$\begin{aligned} \gamma([a, b]) &\subset V, & \bar{q}^i(\gamma(t)) &= 0 & \forall t \in (c, d) \cap \gamma^{-1}(V) \\ \hat{\gamma}([a, b]) &\subset W, & x^\alpha(\hat{\gamma}(t)) &= 0 & \forall t \in (c, d) \cap \hat{\gamma}^{-1}(W) \end{aligned}$$

Without loss of generality we may assume $\pi(W) \subset V$. The restriction to W of the projection $\pi : \mathcal{A} \rightarrow \mathcal{V}_{n+1}$ is then described in coordinates as

$$\bar{q}^i = \bar{q}^i(t, x^1, \dots, x^{n+r})$$

with $\text{rank} \left\| \frac{\partial(\bar{q}^1 \dots \bar{q}^n)}{\partial(x^1 \dots x^{n+r})} \right\| = n$. In particular, the differentials $dt, d\bar{q}^1, \dots, d\bar{q}^n$ are linearly independent everywhere on W .

Let $\mu^A := \mu^A_\alpha(t) dx^\alpha|_{\hat{\gamma}(t)}$ denote r linear differential forms along $\hat{\gamma}$, depending differentiably on t , and completing $dt|_{\hat{\gamma}(t)}, d\bar{q}^i|_{\hat{\gamma}(t)}$ to a basis of $T^*_{\hat{\gamma}(t)}(\mathcal{A})$.

Define r differentiable functions on W by

$$\bar{z}^A = \sum_{\alpha=1}^{n+r} \mu^A_\alpha(t) x^\alpha$$

Then, by construction, the Jacobian $\left\| \frac{\partial(\bar{q}^1 \dots \bar{q}^n \bar{z}^1 \dots \bar{z}^r)}{\partial(x^1 \dots x^{n+r})} \right\|$ is non singular at each point $\hat{\gamma}(t)$. The functions t, \bar{q}^i, \bar{z}^A form therefore a coordinate system in a neighborhood U of the intersection $\hat{\gamma}((c, d)) \cap W$. The system is automatically fibered over \mathcal{V}_{n+1} , and satisfies both properties (B.1a, b), and the first condition (B.1c).

To complete the proof, let $\bar{q}^i = \bar{\psi}^i(t, \bar{q}^i, \bar{z}^A)$ denote the representation of the imbedding $\mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$ in the coordinates t, \bar{q}^i, \bar{z}^A . Under an arbitrary linear transformation $q^i = \alpha^i_j(t) \bar{q}^j$, $z^A = \bar{z}^A$ we have then the transformation laws

$$\psi^i = \frac{d\alpha^i_j}{dt} \bar{q}^j + \alpha^i_j \bar{\psi}^j, \quad \frac{\partial \psi^i}{\partial q^k} = \left(\frac{d\alpha^i_j}{dt} + \alpha^i_r \frac{\partial \bar{\psi}^r}{\partial \bar{q}^j} \right) (\alpha^{-1})^j_k$$

In particular, if the matrix $\alpha^i_j(t)$ is a solution of the differential equation

$$\frac{d\alpha^i_j}{dt} + \alpha^i_r \left(\frac{\partial \bar{\psi}^r}{\partial \bar{q}^j} \right)_{\hat{\gamma}(t)} = 0$$

the coordinates t, q^i, z^A satisfies all stated requirements. \square

Every local chart (U, k) satisfying eqs. (B.1a, b, c) will be said to be *adapted* to the closed arc $(\hat{\gamma}, [a, b])$.

Corollary B.1 *Let $\hat{\gamma} = \{(\hat{\gamma}^{(s)}, [a_{s-1}, a_s]), s = 1, \dots, N\}$ be the lift of an admissible piecewise differentiable section $(\gamma, [t_0, t_1])$. Then, there exist fibered local charts (U_s, k_s) , $k_s = (t, q^1_{(s)}, \dots, q^n_{(s)}, z^1_{(s)}, \dots, z^r_{(s)})$ adapted to the arcs $\hat{\gamma}^{(s)}$ such that, in each intersection $\pi(U_s) \cap \pi(U_{s+1})$, the coordinate transformation $q^i_{(s+1)} = q^i_{(s+1)}(t, q^1_{(s)}, \dots, q^n_{(s)})$ satisfies the condition*

$$\left(\frac{\partial q^i_{(s+1)}}{\partial q^j_{(s)}} \right)_{\gamma(a_s)} = \delta^i_j \quad (\text{B.2})$$

The result follows at once from the proof of Lemma B.1, observing that the adapted coordinates are defined up to arbitrary linear transformations of the form $q'^i = A^i_j q^j$, $z'^A = z^A$, ($A^i_j = \text{const.}$), thus leaving full control on the values of the Jacobians $\left(\frac{\partial q^i_{(s+1)}}{\partial q^j_{(s)}} \right)_{\gamma(a_s)}$ at the corners. \square

Every family $\{(U_s, k_s), s = 1, \dots, N\}$ satisfying the requirements of Corollary B.1 will be said to be *adapted* to the lift $\hat{\gamma}$.

Assigning an adapted family of local charts automatically singles out a distinguished infinitesimal control $h_{(s)}$ along each arc $\gamma^{(s)}$, uniquely defined by the requirement

$$h_{(s)} \left(\frac{\partial}{\partial q^i_{(s)}} \right)_{\gamma^{(s)}(t)} = \left(\frac{\partial}{\partial q^i_{(s)}} \right)_{\hat{\gamma}^{(s)}(t)} \iff h_i^A(t) = 0$$

In view of eqs. (2.34b), (2.35a), (B.1c), the absolute time derivative associated with $h_{(s)}$ is described in coordinates as

$$\frac{D}{Dt} \left(\frac{\partial}{\partial q^i_{(s)}} \right)_{\gamma^{(s)}(t)} = 0 \quad s = 1, \dots, N \quad (\text{B.3})$$

Noting that, by eq. (B.2), the fields $\left(\frac{\partial}{\partial q_{(s)}^i}\right)_{\gamma^{(s)}(t)}$ are continuous at the corners $\gamma(a_s)$ we conclude that the sections $e_{(i)} : [t_0, t_1] \rightarrow V(\gamma)$ given by

$$e_{(i)}(t) = \left(\frac{\partial}{\partial q_{(s)}^i}\right)_{\gamma^{(s)}(t)} \quad \forall t \in [a_{s-1}, a_s], \quad s = 1, \dots, N \quad (\text{B.4})$$

form a basis for the space V_h of h -transported vector fields along γ .

On account of eq. (B.1c), the corresponding dual basis for the space V_h^* is given by $e^{(i)}(t) = \hat{\omega}^i|_{\gamma^{(s)}(t)} = dq_{(s)}^i|_{\gamma^{(s)}(t)} \quad \forall t \in [a_{s-1}, a_s], \quad s = 1, \dots, N$. By definition, together with eqs. (B.3) we have therefore the dual relations

$$\frac{D}{Dt} \hat{\omega}^i|_{\gamma^{(s)}(t)} = 0 \quad (\text{B.5})$$

(ii) Let us now come to the main question. Let $\gamma := \{(\gamma^{(s)}, [a_{s-1}, a_s])\}$ denote an admissible, piecewise differentiable section, $\{(U_s, k_s)\}$ a family of local charts adapted to $\hat{\gamma}$, and $\{e_{(i)}\}, \{e^{(i)}\}$ the corresponding dual bases for the spaces V_h, V_h^* .

We recall that, with the notation of § 2.5, the most general infinitesimal deformation X of γ vanishing at $t = t_0$ is determined by an element $(Y, \underline{\alpha}) \in \mathfrak{W}$, namely by a vertical vector field Y along $\hat{\gamma}$ and by a collection of real numbers $\underline{\alpha} = (\alpha_1, \dots, \alpha_{N-1})$. In particular, a necessary and sufficient condition for X to satisfy $X(t_1) = 0$ is expressed by the requirement (2.53) which, in adapted coordinates, reads

$$\int_{t_0}^{t_1} Y^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt - \sum_{s=1}^{N-1} \alpha_s [\psi^i(\hat{\gamma})]_{a_s} = 0 \quad (\text{B.6})$$

To inquire whether a given infinitesimal deformation vanishing at the end points of γ is tangent to a finite deformation with fixed end points we introduce an auxiliary tool, namely a positive metric on V_h , described by a symmetric tensor $\Phi = g_{ij} e^{(i)} \otimes e^{(j)}$.

In view of the identification $V(\gamma) \simeq [t_0, t_1] \times V_h$, assigning Φ automatically induces a scalar product along the fibres of $V(\gamma)$. This, in turn, determines a scalar product between vertical vector fields along $\hat{\gamma}$, based on the identification

$$(Y, Z) := (\hat{\rho}(Y), \hat{\rho}(Z)) \quad (\text{B.7})$$

$\hat{\rho} : V(\hat{\gamma}) \rightarrow V(\gamma)$ denoting the homomorphism (2.16). In adapted coordinates, eqs. (2.17), (B.7) provide the evaluation $(Y, Z) = G_{AB} Y^A Z^B$, with

$$G_{AB} = \left(\hat{\rho} \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}, \hat{\rho} \left(\frac{\partial}{\partial z^B} \right)_{\hat{\gamma}} \right) = g_{ij} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} \quad (\text{B.8})$$

As usual, the inverse of the matrix G_{AB} will be denoted by G^{AB} .

By means of Φ , to every $\underline{\alpha} = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbb{R}^{N-1}$ we associate $N - 1$ functions $a_s(\xi)$ according to the prescription

$$a_s(\xi) := a_s + \alpha_s \xi - \frac{1}{2} \alpha_s^2 \xi^2 g_{ij} \nu^i [\psi^j(\hat{\gamma})]_{a_s} \quad s = 1, \dots, N-1 \quad (\text{B.9})$$

For notational convenience, the family is completed by the constant functions $a_0(\xi) = t_0$, $a_N(\xi) = t_1$.

In a similar way, given any vertical vector field Y along $\hat{\gamma}$, meant as a family of fields $Y_{(s)} = Y_{(s)}^A \left(\frac{\partial}{\partial z^A} \right)_{\hat{\gamma}(s)}$ along the arcs of $\hat{\gamma}$, for each $\nu \in V_h$ we denote by $\sigma_{(\xi, \nu)}^{(s)} : \pi(U_s) \rightarrow U_s$, $s = 1, \dots, N$ the $(n + 1)$ -parameter families of sections described in coordinates as

$$z_{(s)}^A = \xi Y_{(s)}^A(t) + \frac{1}{2} \xi^2 \chi_{(s)i}^A(t) \nu^i \quad (\text{B.10})$$

with

$$\chi_{(s)i}^A(t) := g_{ik} G^{AB} \left(\frac{\partial \psi^k}{\partial z^B} \right)_{\hat{\gamma}} \quad (\text{B.11})$$

It goes without saying that, being strictly coordinate-dependent, eq. (B.10) has no intrinsic geometrical meaning, but is merely a technical tool, whose usefulness will be clear in the subsequent discussion.

On account of eqs. (B.10), (B.11) it is easily seen that, given any open subset $A \subset V_h$ with compact closure, there exists $m > 0$ such that the image $\sigma_{(\xi, \nu)}^{(s)}(\pi(U_s))$ is entirely contained in U_s for all $\nu \in A$, $|\xi| < m$, $s = 1, \dots, N$.

Theorem B.1 *Let γ be an admissible, piecewise differentiable evolution, and $(Y, \underline{\alpha})$ an admissible infinitesimal deformation of γ vanishing at the endpoints. Define the metric Φ and the functions $\chi_{(s)i}^A(t)$, $a_s(\xi)$ as above. Then, given any open subset $\Delta \subset V_h$ with compact closure there exist an $\varepsilon > 0$ and a family $\gamma_{(\xi, \nu)} = \{(\gamma_{(\xi, \nu)}^{(s)}, [a_{s-1}(\xi), a_s(\xi)])\}$ of piecewise differentiable admissible sections defined for $|\xi| < \varepsilon$, $\nu \in \Delta$ and satisfying the properties*

- a) $\gamma_{(0, \nu)}(t) = \gamma(t) \quad \forall \nu;$
- b) $\gamma_{(\xi, \nu)}(t_0) = \gamma(t_0) \quad \forall \xi, \nu;$
- c) $\gamma_{(\xi, \nu)}^{(s)}(a_s(\xi)) = \gamma_{(\xi, \nu)}^{(s+1)}(a_s(\xi)) \quad \forall s = 1, \dots, N-1$
- d) each arc $\gamma_{(\xi, \nu)}^{(s)}(t)$, expressed in coordinates as $q_{(s)}^i = \varphi_{(s)}^i(t, \xi, \nu^i)$, satisfies the control equation

$$\frac{\partial \varphi_{(s)}^i}{\partial t} = \psi^i \left(t, \varphi_{(s)}^i, \xi Y_{(s)}^A(t) + \frac{1}{2} \xi^2 \chi_{(s)i}^A \nu^i \right) \quad (\text{B.12})$$

Proof. Let $A \subset V_h$ denote an open set with compact closure containing $\bar{\Delta}$.

We choose $m \in \mathbb{R}_+$ as above, and examine the situation separately in each adapted chart (U_s, k_s) .

There, solving eq. (B.12) amounts to determining the integral curves of the $(n+1)$ -parameter family of vector fields $Z_{(\xi, \nu)}^{(s)} = \frac{\partial}{\partial t} + Z_{(s)}^i \frac{\partial}{\partial q^i}$ on $\pi(U_s)$, with $Z_{(s)}^i = \psi^i(t, q^k, \xi Y_{(s)}^A(t) + \frac{1}{2} \xi^2 \chi_{(s)h}^A(t) \nu^h)$.

This, in turn, is equivalent to determining the integral curves of a single vector field $\tilde{Z}_{(s)} = \frac{\partial}{\partial t} + Z_{(s)}^i \frac{\partial}{\partial q^i}$ in the product manifold $(-m, m) \times A \times \pi(U_s)$.

Let $\zeta_{(\xi, \nu)}^{(s)}(t, x)$ denote the integral curve of $\tilde{Z}_{(s)}$ through the point (ξ, ν, x) . Also, let x_{s-1} denote the corner $\gamma(a_{s-1})$. Then, on account of eq. (B.1c), for any $\nu^* \in A$ the curve $\zeta_{(0, \nu^*)}^{(s)}(t, x_{s-1})$ coincides with the coordinate line $q^i = 0, \xi = 0, \nu = \nu^*$, and is therefore defined for all t in an open interval $(b_{s-1}, b_s) \supset [a_{s-1}, a_s]$.

By well known theorems in ordinary differential equations [13, 3] this implies the existence of an open neighborhood $W_{s-1} \ni (0, \nu^*, x_{s-1})$ such that the curve $\zeta_{(\xi, \nu)}^{(s)}(t, x)$ is defined for all $(\xi, \nu, x) \in W_{s-1}$ and all t in the closed interval $[t(x), a_s(\xi)] \subset (b_{s-1}, b_s)$.

In particular, denoting by Σ_s the slice $t = a_s(\xi)$ in $(-m, m) \times A \times \pi(U_s)$, we conclude that the 1-parameter group of diffeomorphisms determined by the field $\tilde{Z}_{(s)}$ maps the intersection $W_{s-1} \cap \Sigma_{s-1}$ into an open neighborhood of the point $(0, \nu^*, x_s)$ in Σ_s . Without loss of generality we may always arrange that the image of each $W_{s-1} \cap \Sigma_{s-1}$ is contained in $W_s \cap \Sigma_s$, $s = 1, \dots, N$.

The rest is now entirely straightforward: let U and $\varepsilon_U > 0$ respectively denote an open neighborhood of ν^* in A and a positive number such that $(\xi, \nu, x_0) \in W_0 \cap \Sigma_0 \forall |\xi| < \varepsilon_U, \nu \in U$ (notice that, according to our thesis, we are “freezing” the choice of the point x_0). For each $|\xi| < \varepsilon_U, \nu \in U$ consider the sequence of closed arcs $\gamma_{(\xi, \nu)}^{(s)} : [a_{s-1}(\xi), a_s(\xi)] \rightarrow \pi(U_s)$ defined inductively by

$$\begin{aligned} \gamma_{(\xi, \nu)}^{(1)}(t) &= \zeta_{(\xi, \nu)}^{(1)}(t, x_0) & t \in [t_0, a_1(\xi)] \\ \gamma_{(\xi)}^{(s+1)}(t) &= \zeta_{(\xi, \nu)}^{(s+1)}(t, \gamma_{(\xi)}^{(s)}(a_s(\xi))) & t \in [a_s(\xi), a_{s+1}(\xi)] \end{aligned}$$

The collection $\gamma_{(\xi, \nu)} := \{(\gamma_{(\xi, \nu)}^{(s)}, [a_{s-1}(\xi), a_s(\xi)]), s = 1, \dots, N\}$ is then easily recognized to define an $(n+1)$ -parameter family of continuous, piecewise differentiable sections satisfying all requirements a), b), c), d) of the Theorem.

To complete our proof let us finally recall that, for any $\nu^* \in A$, the family $\gamma_{(\xi, \nu)}$ exists for all ν in an open neighborhood $U \ni \nu^*$ and all $|\xi| < \varepsilon_U$. On the other hand, by the assumed compactness of $\bar{\Delta}$, the subset $\Delta \subset A$ may be covered by a finite number of subsets $\{U_1, \dots, U_k\}$ of the required type.

This establishes Theorem B.1, with $\varepsilon = \min\{\varepsilon_{U_1}, \dots, \varepsilon_{U_k}\}$. \square

According to Theorem B.1, for any open subset $\Delta \subset V_h$ with compact closure, the correspondence $\nu \rightarrow \gamma_{(\xi, \nu)}(t_1)$ sets up a 1-parameter family of differentiable maps of Δ into the hypersurface $t = t_1$, with values in a neighborhood

of the point $\gamma(t_1)$. Moreover, given any differentiable curve $\nu = \nu(\xi)$ in Δ , the 1-parameter family of sections $\gamma_{(\xi, \nu(\xi))}(t)$, $|\xi| < \varepsilon$, $t \in [t_0, t_1]$ is a deformation of γ tangent to the original infinitesimal deformation X determined by $(Y, \alpha_1, \dots, \alpha_{N-1})$, and leaving the first end point x_0 fixed.

Our original problem is therefore reduced to establishing the existence of a curve $\nu(\xi)$ satisfying $\gamma_{(\xi, \nu(\xi))}(t_1) \equiv \gamma(t_1)$ in some open neighborhood of $\xi = 0$.

In adapted coordinates, setting for simplicity $\varphi^i(\xi, \nu) := \varphi_{(N)}^i(t_1, \xi, \nu)$, the required condition reads

$$\varphi^i(\xi, \nu^1, \dots, \nu^n) = 0 \quad i = 1, \dots, n \quad (\text{B.13a})$$

Taking the relations $\varphi^i(0, \nu) = q_{(N)}^i(\gamma(t_1)) = 0$, $(\frac{\partial \varphi^i}{\partial \xi})_{\xi=0} = X^i(t_1)$ into account, a straightforward application of Taylor's theorem shows that, whenever the condition $X(t_1) = 0$ holds true, i.e. whenever the field Y and the coefficients α_s satisfy eq. (B.6), the functions φ^i are necessarily of the form $\varphi^i(\xi, \nu) = \xi^2 \theta^i(\xi, \nu)$, with $\theta^i(\xi, \nu)$ regular at $\xi = 0$. Under the stated assumptions, eq. (B.13a) is therefore equivalent to the condition

$$\theta^i(\xi, \nu^1, \dots, \nu^n) = 0 \quad i = 1, \dots, n \quad (\text{B.13b})$$

Let us discuss the solvability of eqs. (B.13b) for the ν^i 's as functions of ξ in a neighborhood of $\xi = 0$. To this end we observe that the matching conditions c) of Theorem 3.1 give rise to relations of the form

$$\varphi_{(s+1)}^i(a_s(\xi), \xi, \nu) = q_{(s+1)}^i(a_s(\xi), \varphi_{(s)}^1(a_s(\xi), \xi, \nu), \dots, \varphi_{(s)}^n(a_s(\xi), \xi, \nu))$$

$q_{(s+1)}^i = q_{(s+1)}^i(t, q_{(s)}^1, \dots, q_{(s)}^n)$ denoting the transformation between adapted coordinates in the intersection $\pi(U_s \cap U_{s+1})$. From these, deriving with respect to ξ we get the expressions

$$\frac{\partial \varphi_{(s+1)}^i}{\partial t} \frac{da_s}{d\xi} + \frac{\partial \varphi_{(s+1)}^i}{\partial \xi} = \frac{\partial q_{(s+1)}^i}{\partial t} \frac{da_s}{d\xi} + \frac{\partial q_{(s+1)}^i}{\partial q_{(s)}^k} \left(\frac{\partial \varphi_{(s)}^k}{\partial t} \frac{da_s}{d\xi} + \frac{\partial \varphi_{(s)}^k}{\partial \xi} \right) \quad (\text{B.14})$$

At $\xi = 0$, recalling eqs. (2.44), (B.2), (B.9) as well as the identification $X_{(s)}^i = \frac{\partial \varphi_{(s)}^i}{\partial \xi} \Big|_{\xi=0}$ the latter provide the relation

$$X_{(s+1)}^i(a_s) = \alpha_s \frac{\partial q_{(s+1)}^i}{\partial t} \Big|_{x_s} + X_{(s)}^i(a_s) \Rightarrow \frac{\partial q_{(s+1)}^i}{\partial t} \Big|_{x_s} = - \left[\psi^i(\hat{\gamma}) \right]_{a_s} \quad (\text{B.15})$$

In a similar way, on account of eqs. (B.2), (B.9), (B.15), deriving eq. (B.14) with respect to ξ and evaluating everything at $\xi = 0$, a straightforward calculation

yields the result

$$\begin{aligned}
\left[\frac{\partial^2 \varphi_{(s+1)}^i}{\partial \xi^2} - \frac{\partial^2 \varphi_{(s)}^i}{\partial \xi^2} \right]_{x_s} &= \\
&= \alpha_s^2 \frac{\partial^2 q_{(s+1)}^i}{\partial t^2} + 2\alpha_s \frac{\partial^2 q_{(s+1)}^i}{\partial t \partial q_{(s)}^k} X_{(s)}^k + \frac{\partial^2 q_{(s+1)}^i}{\partial q_{(s)}^h \partial q_{(s)}^k} X_{(s)}^h X_{(s)}^k - \\
&- 2\alpha_s \left[\frac{dX_{(s+1)}^i}{dt} - \frac{dX_{(s)}^i}{dt} \right]_{x_s} + \alpha_s^2 \left[\psi^i(\hat{\gamma}) \right]_{a_s} g_{rk} \left[\psi^r(\hat{\gamma}) \right]_{a_s} \nu^k \quad (\text{B.16})
\end{aligned}$$

expressing the jumps $\left[\frac{\partial^2 \varphi_{(s+1)}^i}{\partial \xi^2} - \frac{\partial^2 \varphi_{(s)}^i}{\partial \xi^2} \right]_{x_s}$ in terms of the section γ , of the infinitesimal deformation and of the variables ν^i .

In addition to this let us now make use of the fact that, in each adapted chart, eqs. (B.12) imply the evolution equations

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{\partial^2 \varphi_{(s)}^i}{\partial \xi^2} \right)_{\xi=0} &= \left(\frac{\partial^2 \psi^i}{\partial q^k \partial q^r} \right)_{\hat{\gamma}(s)} X^k X^r + 2 \left(\frac{\partial^2 \psi^i}{\partial q^k \partial z^A} \right)_{\hat{\gamma}(s)} X^k Y^A + \\
&+ \left(\frac{\partial^2 \psi^i}{\partial z^A \partial z^B} \right)_{\hat{\gamma}(s)} Y^A Y^B + \cancel{\left(\frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}(s)} \left(\frac{\partial^2 \varphi_{(s)}^k}{\partial \xi^2} \right)_{\xi=0}} + \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}(s)} \chi_{(s)k}^A \nu^k
\end{aligned}$$

the cancelation arising from eq. (B.1c).

From the latter, restoring the notation $\varphi^i(\xi, \nu)$ for $\varphi_{(N)}^i(t_1, \xi, \nu)$ and recalling eqs. (B.11), (B.16), as well as the fact that the components g_{ij} are by definition *constant* along γ , we get an expression of the form

$$\begin{aligned}
\theta^i|_{\xi=0} &= \left(\frac{\partial^2 \varphi^i}{\partial \xi^2} \right)_{\xi=0} = \\
&= b^i + \left(\sum_{s=1}^N \int_{a_{s-1}}^{a_s} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}(s)(t)} \chi_{(s)k}^A(t) dt + \sum_{s=1}^{N-1} \alpha_s^2 \left[\psi^i(\hat{\gamma}) \right]_{a_s} g_{jk} \left[\psi^j(\hat{\gamma}) \right]_{a_s} \right) \nu^k = \\
&= b^i + \left(\int_{t_0}^{t_1} G^{AB} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} dt + \sum_{s=1}^{N-1} \alpha_s^2 \left[\psi^i(\hat{\gamma}) \right]_{a_s} \left[\psi^j(\hat{\gamma}) \right]_{a_s} \right) g_{jk} \nu^k \quad (\text{B.17})
\end{aligned}$$

with $b^i \in \mathbb{R}$ depending only on the section γ and on the original infinitesimal deformation. Collecting all results we can therefore state

Proposition B.1 *Let $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ be a continuous, piecewise differentiable, admissible section. Then, if the matrix*

$$S^{ij} := \int_{t_0}^{t_1} G^{AB} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} dt + \sum_{s=1}^{N-1} \alpha_s^2 \left[\psi^i(\hat{\gamma}) \right]_{a_s} \left[\psi^j(\hat{\gamma}) \right]_{a_s} \quad (\text{B.18})$$

is non-singular, every infinitesimal deformation of γ vanishing at the end points is tangent to a finite deformation with fixed end points.

Proof. The conclusion follows at once from the fact that, on account of eq. (B.17), the non-singularity of the matrix (B.18) ensures the solvability of eqs. (B.13b) in a neighborhood of $\xi = 0$. \square

Proposition B.1 may be rephrased in the language of § 2.5 observing that, whenever the section γ is abnormal, the matrix (B.18) is necessarily *singular*.

Under the stated circumstance, in fact, Proposition 2.5 and eq. (B.4) imply the existence of at least one non-zero virtual 1-form $\rho_i \hat{\omega}^i|_\gamma$ with *constant* components ρ_i obeying the relations

$$\rho_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}(t)} = 0, \quad \rho_i [\psi^i(\hat{\gamma})]_{a_s} = 0 \quad (\text{B.19})$$

and therefore automatically satisfying $\rho_i S^{ij} = 0$.

More specifically, denoting by p the *abnormality index* of γ , we have the following

Theorem B.2 *The matrix (B.18) has rank $n - p$.*

Proof. By definition, the index p coincides with the dimension of the annihilator $(\Upsilon(\mathfrak{W}))^0 \subset V_h^*$, which, in turn, is identical to the dimension of the space of constant solutions of eqs. (B.19).

On the other hand, by eqs. (B.8), (B.18), the matrix S^{ij} is positive semidefinite. Its kernel is therefore identical to the totality of zeroes of the quadratic form $S^{ij} \rho_i \rho_j$, i.e. to the totality of n -tuples $(\rho_1, \dots, \rho_n) \in \mathbb{R}^n$ satisfying the relation

$$\begin{aligned} 0 &= \left(\int_{t_0}^{t_1} G^{AB} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} dt + \sum_{s=1}^{N-1} \alpha_s^2 [\psi^i(\hat{\gamma})]_{a_s} [\psi^j(\hat{\gamma})]_{a_s} \right) \rho_i \rho_j = \\ &= \int_{t_0}^{t_1} G^{AB} \left(\rho_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \right) \left(\rho_j \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} \right) dt + \sum_{s=1}^{N-1} \alpha_s^2 \left(\rho_i [\psi^i(\hat{\gamma})]_{a_s} \right)^2 \end{aligned}$$

Due to the positive definiteness of $G^{AB}(t)$, the last condition is equivalent to eqs. (B.19). This proves $\dim(\ker(S^{ij})) = p \Rightarrow \text{rank}(S^{ij}) = n - p$. \square

In the language of § 2.4, Proposition B.1 and Theorem B.2 show that the normal evolutions form a subset of the ordinary ones, thus establishing Proposition 2.6.

Along the same lines, a deeper result is provided by the following

Theorem B.3 *Let p (≥ 0) denote the abnormality index of γ . Then a sufficient condition for every infinitesimal deformation vanishing at the end points of γ to be tangent to a finite deformation with fixed end points is the existence of an $(n-p)$ -dimensional submanifold $S \subset \mathcal{V}_{n+1}$ contained in the slice $t = t_1$ and containing the point $\gamma(t_1)$, such that every deformation γ_ξ leaving $\gamma(t_0)$ fixed satisfies $\gamma_\xi(t_1) \in S$ for all ξ sufficiently small.*

Proof. Assuming the existence of a submanifold $S \xrightarrow{i} \mathcal{V}_{n+1}$ with the stated properties, we denote by $(V, \zeta^1, \dots, \zeta^{n-p})$ a local chart in S centered at the point $\gamma(t_1)$, and by

$$t = t_1, \quad q_{(N)}^i = \varrho^i(\zeta^1, \dots, \zeta^{n-p}) \quad (\text{B.20})$$

the representation of S in adapted coordinates.

Then, for any open subset $\Delta \subset V_h$ with compact closure and for sufficiently small ξ , the correspondence $(\xi, \nu) \rightarrow \gamma_{(\xi, \nu)}(t_1)$ factors through S , giving rise to a differentiable map $g : (-\varepsilon, \varepsilon) \times \Delta \rightarrow S$ satisfying the relation $\gamma_{(\xi, \nu)}(t_1) = i \cdot g(\xi, \nu)$.

In coordinates, setting $\zeta^\alpha(g(\xi, \nu)) = g^\alpha(\xi, \nu^1, \dots, \nu^n)$ and resuming the notation $\varphi^i(\xi, \nu^1, \dots, \nu^n)$ for $q_{(N)}^i(\gamma_{(\xi, \nu)}(t_1))$, this provides the identification

$$\varphi^i(\xi, \nu^1, \dots, \nu^n) = \varrho^i(g^1(\xi, \nu^1, \dots, \nu^n), \dots, g^{n-p}(\xi, \nu^1, \dots, \nu^n)) \quad (\text{B.21})$$

From the latter, recalling the relation $g^\alpha(0, \nu^1, \dots, \nu^n) = \zeta^\alpha(\gamma(t_1)) = 0$ as well as the fact that the Jacobian $\frac{\partial(\varrho^1 \dots \varrho^n)}{\partial(\zeta^1 \dots \zeta^{n-p})}$ has maximal rank it is easily seen that the equalities $\varphi^i(0, \nu^1, \dots, \nu^n) = \frac{\partial \varphi^i}{\partial \xi}(0, \nu^1, \dots, \nu^n) = 0$ are reflected into analogous properties of the functions g^α .

By Taylor's theorem we have therefore an expression of the form

$$g^\alpha = \xi^2 \mu^\alpha(\xi, \nu^1, \dots, \nu^n) \quad (\text{B.22})$$

with the functions μ^α regular at $\xi = 0$.

The proof is thus reduced to establishing the solvability of the system

$$\mu^\alpha(\xi, \nu^1, \dots, \nu^n) = 0 \quad (\text{B.23})$$

for the ν^i 's as functions of ξ in a neighborhood of $\xi = 0$.

To this end, by direct computation, from eqs. (B.21), (B.22) we derive the relation

$$\left(\frac{\partial^2 \varphi^i}{\partial \xi^2} \right)_{\xi=0} = 2 \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha} \right)_{(0, \dots, 0)} \mu^\alpha \big|_{\xi=0} = 2 \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha} \right)_{\gamma(t_1)} \mu^\alpha(0, \nu^1, \dots, \nu^n)$$

Together with eqs. (B.17), (B.18), the latter provides the identification

$$b^i + S^{ir} g_{rk} \nu^k = 2 \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha} \right)_{\gamma(t_1)} \mu^\alpha(0, \nu^1, \dots, \nu^n) \quad (\text{B.24})$$

In view of eq. (B.24), the functions $\mu^\alpha(0, \nu^1, \dots, \nu^n)$ are linear polynomials

$$\mu^\alpha(0, \nu^1, \dots, \nu^n) = M^\alpha_k \nu^k + c^\alpha \quad (\text{B.25})$$

with coefficients M^α_k , c^α uniquely determined in terms of b^i , S^{ir} , g_{rk} and of the imbedding (B.20). In particular, by eq. (B.24), the rank of the matrix M^α_k cannot be smaller than the rank of S^{ij} and, of course, cannot exceed $n - p$. According to Theorem B.2, we have therefore $\text{rank } M^\alpha_k = n - p$.

Collecting all results we conclude:

- the system (B.23) admits ∞^p solutions of the form $(0, \nu^{*1}, \dots, \nu^{*n})$;
- on account of eq. (B.25), the Jacobian $\left\| \frac{\partial(\mu^1 \dots \mu^{n-p})}{\partial(\nu^1 \dots \nu^n)} \right\|$ has rank $n - p$ at each point $(0, \nu^1, \dots, \nu^n)$. By continuity, it has therefore rank $n - p$ in a neighborhood of every solution $(0, \nu^{*1}, \dots, \nu^{*n})$ of eqs. (B.23).

By the implicit function theorem, this proves that the system (B.23) admits at least a solution of the form $\nu^i = \nu^i(\xi)$ in a neighborhood of $\xi = 0$ (actually, infinitely many solutions whenever $p > 0$). \square

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